Localization on D-brane and Gauge Theory/Matrix Model

Kazutoshi Ohta

Theoretical Physics Laboratory, RIKEN

T.Matsuo, S.Matsuura and KO, hep-th/0406191
S.Matsuura and KO, work in progress
Introduction

Holomorphic quantity of supersymmetric gauge theory is obtained from ‘bosonic’ theory and integrable systems.

Example

- Topological string on $\text{CY}_3$
  - 3D Chern-Simons gauge theory on 3-cycle [Gopakumar-Vafa]

- Effective superpotential of $N=1$ supersymmetric gauge theory
  - holomorphic matrix model [Dijkgraaf-Vafa]

- Instanton corrections to prepotential of $N=2$ supersymmetric gauge theory
  - random partitions [Nekrasov]
Why are they bosonic?

Effective theory on D-brane wrapping internal cycles (Euclidean D-brane) should be topologically twisted. [Bershadsky-Sadov-Vafa]

Localization reduces topologically twisted theory to bosonic one.

D-strings wrapping on 2-cycle
= 4D instantons

Localization

(Generalized) 2D Yang-Mills

Large $N$

Nekrasov’s partition function
Overview of today’s talk:

D-branes wrapping 2-cycle

Localization

Discrete Matrix Model

Ensemble of instantons (Nekrasov)

D-branes wrapping 2-cycle

Bosonic 2D YM theory

Discrete Matrix Model

Dijkgraaf-Vafa

Random partitions

T-dual

Large N

Continuous

Discrete Unitary Matrix Model

Bosonic 3D CS gauge theory

$\begin{align*}
\text{c=1 Matrix Model} \\
\text{c=0 Matrix Model}
\end{align*}$

- Topological string
- BPS state counting in 5D gauge theory
Localization on D-brane

- Topological field theory on D-brane

We consider D5-brane wrapping on 2-cycle in the internal ALE \((T^* \Sigma_G)\) surface:

Target space \(\mathbb{R}^{1,3} \times \mathbb{C} \times \mathcal{M}_4\)

Worldvolume of D5 \(\mathbb{R}^{1,3} \times \Sigma_G\)

\[ \begin{align*}
\mathcal{M}_4 & \quad \text{Genus } G \\
\Sigma_G & \quad \text{Riemann surface}
\end{align*} \]

Area

\[ \frac{1}{g_{YM}^2} \propto \frac{A}{g_s l_s^2} \]
BRST transformations

\[ QA = \lambda, \quad Q\lambda = -d_A \Phi, \]
\[ Q\Phi = 0, \]
\[ Q\Phi = \eta, \quad Q\eta = [\Phi, \Phi], \]
\[ QH = [\Phi, \chi], \quad Q\chi = H. \]

BRST exact action

\[ S = \frac{1}{g_s} Q \int_{\Sigma_G} d^2z \text{Tr} \, \Xi(A, \lambda, \Phi, \Phi, \eta, H, \chi) \]

where

\[ \Xi = \frac{1}{4} \eta[\Phi, \Phi] + \chi(*F - H) + \frac{1}{2} \lambda \wedge d_A \Phi \]
Roughly speaking, the partition function gives a “volume” of the flat connection moduli space

\[
Z_{\text{top}} = \int D\tilde{B} D\tilde{F} D\Phi e^{-\frac{1}{g_s} Q \int G d^2 z \text{Tr} \Xi(B,F,\Phi)} \\
\simeq \int_{\mathcal{M}_{F=0}} \text{“1”} = \text{“Vol}(\mathcal{M}_{F=0})\text{”}
\]

since the partition function is independent of the coupling and localized on the flat connections

\[
\text{cf.} \quad Z_{\text{Nekrasov}}^k = Z_{\text{D-instanton}}^k \simeq \int_{\mathcal{M}_{r,k}} \text{“1”}
\]

0-dim. reduced matrix model from 6d $N=1$ Yang-Mills
Observable (chemical potential for D0)

\[ I_2(\Sigma_G) = \int_{\Sigma_G} \text{Tr} \left( i\Phi F + \frac{1}{2} \lambda \wedge \lambda \right) \]

Vev of the observable is equivalent to a bosonic physical BF theory partition function [Witten]

\[ \left\langle e^{-\frac{1}{g_s} I_2(\Sigma_G)} \right\rangle_{\text{top}} = \int \mathcal{D}A\mathcal{D}\Phi \exp \left[ -\frac{1}{g_s} \int_{\Sigma_G} \text{Tr} i\Phi F \right] \]

since if we integrate out \( \Phi \), the partition function becomes

\[ \int \mathcal{D}A\delta(F) = \text{“Vol}(\mathcal{M}_{F=0})” \]
- We can deform the theory by adding a potential for $\Phi$

$$\text{Tr } W(\Phi) = \sum_{k=1}^{r+1} \frac{\mu_k}{k} \text{Tr } \Phi^k$$

- The following partition functions are NOT coincides with each other generally

$$\left\langle \exp \left[ -\frac{1}{g_s} \int_{\Sigma_G} d^2 z \text{Tr} \left( i\Phi F + \lambda \wedge \lambda + W(\Phi) \right) \right] \right\rangle_{\text{top}}$$

$$\neq \int \mathcal{D}A\mathcal{D}\Phi \exp \left[ -\frac{1}{g_s} \int_{\Sigma_G} d^2 z \text{Tr} \left( i\Phi F + W(\Phi) \right) \right]$$

2D analog to the Gromov-Witten invariant
However, if we consider only near critical points, the bosonic theory evaluation is good approximation, since eq. of motion says

$$iF = W'(\Phi)$$

BF theory with a quadratic potential gives a ordinary 2D Yang-Mills theory

$$\frac{1}{g_s} \int d^2 z \text{Tr} (i\Phi F + \frac{\mu}{2} \Phi^2) \Rightarrow \frac{1}{2g_{YM}^2} \int d^2 z F^2$$

For general $W(\Phi)$ generalized 2D YM
Migdal’s partition function (Gaussian model)

- We consider the partition function of the BF theory with the quadratic potential

\[ Z = \int \mathcal{D}A \mathcal{D}\Phi \exp \left[ -\frac{1}{g_s} \int_{\Sigma_G} d^2z \text{Tr} \left( i\Phi F + \frac{\mu}{2} \Phi^2 \right) \right] \]

Gauge fix: \( \Phi \rightarrow \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \)

Integrating out ghosts

Summing up possible topological sector

\[ \left( \int_{\Sigma_G} dA_i = 2\pi p_i, \quad p_i \in \mathbb{Z} \right) \]

\[ Z = \sum_{p_k \in \mathbb{Z}} \int d\lambda_k \prod_{i<j} (\lambda_i - \lambda_j)^2 - 2G \prod_k \exp \left[ -\frac{1}{g_s} \left( 2\pi i \lambda_k p_k + \frac{\mu A}{2} \lambda_k^2 \right) \right] \]
Integrating out $\lambda_i$, we obtain a discrete MM

$$
Z = N! \sum_{n_1 > n_2 > \cdots > n_N} \prod_{i<j} (g_s n_i - g_s n_j)^2 \exp \left[ -\frac{g_s \mu A}{2} \sum_i n_i^2 \right]
$$

where we use

$$
\sum_{p_i \in \mathbb{Z}} \exp \left( -2\pi i \frac{\lambda_i}{g_s} p_i \right) = \sum_{n_i \in \mathbb{Z}} \delta(\lambda_i / g_s - n_i)
$$

- Strongly decreasing integer sequence

$$
n_1 > n_2 > \cdots > n_N
$$

can be rewritten in terms of weakly decreasing sequence through

$$
n_i = k_i - i + \frac{N+1}{2}
$$

Young diagram
Melting edge of crystal

Order parameter $\lambda A$ (inverse temp.)

Potential

Edge of crystal at zero temp.

Young Diagram with $k_i$

$N$ free fermions at $x=n_i$

Fermi surface
Monte-Carlo simulation ($N=100$)

$A > A_c$

$A < A_c$

Young diagram

Eigenvalue (fermion) density

Douglas-Kazakov phase transition ($3^{rd}$ order)
The partition function reduces to

\[
\tilde{Z} = \sum_{\{k_i\}} \prod_{i<j} \left( \frac{k_i - k_j - i + j}{i - j} \right)^{2-2G} \exp \left[ -\frac{g_s \mu A}{2} \left( \sum_i (N k_i + k_i(k_i - 2i + 1)) \right) \right]
\]

\[
= \sum_R \text{dim} R^{2-2G} e^{-\frac{\lambda A}{2N} C_2(R)}
\]

where \( \tilde{Z} = Z/Z_0 \) and \( Z_0 = Z(\forall k_i = 0) \)

\[\xrightarrow{\phantom{\text{Vol}(U(N))}} \prod_{j=1}^N j! \approx \text{Vol}(U(N))\]
Nekrasov’s partition function

- We consider the generalized 2D YM partition function

\[ Z = \int \mathcal{D} A \mathcal{D} \Phi \exp \left[ -\frac{1}{g_s} \int \Sigma_G \text{Tr} \left( i \Phi F + W(\Phi) \right) \right] \]

- We obtain the discrete random matrix model in the similar way

\[ Z = \sum_{\{n_i\}} \prod_{1 \leq i < j \leq N} \left( g_s n_i - g_s n_j \right)^{2-2G} e^{-\frac{\lambda A}{g_s N} \sum_i W(g_s n_i)} \]
Summation over possible integers dominates around the critical points of the potential

\[ W'(x) = \mu_r + 1 \prod_{l=1}^{r} (x - a_l) \]

We assume

\[ n_i \to a_l + g_s(k_i^{(l)} - i + \frac{N_l + 1}{2}) \]

The normalized partition function dominated near the critical points

\[
\tilde{Z} = \sum_{l} \prod_{(l,i) \neq (n,j)} \left( \frac{a_l - a_n + g_s(k_i^{(l)} - k_j^{(n)} + j - i + \frac{N_l - N_n}{2})}{a_l - a_n + g_s(j - i + \frac{N_l - N_n}{2})} \right)^{1-G} \\
\times \exp \left[ -\frac{g_s A \mu_r + 1}{2} \sum_l \Delta_l C_2(R^{(l)}) \right]
\]
Monte-Carlo for two-cut solution

Gross-Witten phase transition

Douglas-Kazakov phase transition
Zero temperature limit ($\lambda A \to \infty$)

The “ground state”

contributes to the perturbative part of the
prepotential including the graviphoton correction

$$\mathcal{F}^\text{pert}(a_l) = \sum_{l \neq n} \gamma_{gs}(a_l - a_n)$$

where

$$\gamma_{gs}(x) = \frac{1}{g_s^2} \left( \frac{1}{2} x^2 \log x - \frac{3}{4} x^2 \right) - \frac{1}{12} \log x + \sum_{g=2}^{\infty} \frac{\ln(B_{2g})}{2g(2g-2)} \left( \frac{g_s}{x} \right)^{2g-2}$$
In the large N limit with fixing

$$\mu_r + 1 |\Delta_l| N_l = \lambda$$

we obtain

$$\tilde{Z}_{gYM_2} = |Z_{\text{inst}}|^2$$

where

$$Z_{\text{inst}} = \sum_{\tilde{k}} \prod_{(l,i) \neq (n,j)} \left( \frac{a_i - a_n + g_s(k_i^{(l)} - k_j^{(n)} + j - i)}{a_i - a_n + g_s(j - i)} \right)^{1-G} \Lambda^{2rk}$$

$$\tilde{Z}_{\text{inst}} = \sum_{\tilde{k}} \prod_{(l,i) \neq (n,j)} \left( \frac{\tilde{a}_i - \tilde{a}_n + g_s(-\tilde{k}_i^{(l)} + \tilde{k}_j^{(n)} + j - i)}{\tilde{a}_i - \tilde{a}_n + g_s(j - i)} \right)^{1-G} \tilde{\Lambda}^{2rk}$$
Gross-Taylor chiral decomposition
We take a continuous limit of the discrete matrix model

\[ Z = \sum_{\{n_i\}} \prod_{1 \leq i < j \leq N} \left( g_s n_i - g_s n_j \right)^{2-2G} e^{-\frac{A}{g_s} \sum_i W(g_s n_i)} \]

\[ g_s \to 0, \quad A \to 0 \]

\[ Z = \int \prod_i d\lambda_i \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)^{2-2G} e^{-\frac{1}{g_s} \sum_i W(\lambda_i)} \]
We can reproduce the Dijkgraaf-Vafa’s results from the continuous limit via

\[ W_{\text{eff}} = \sum_{i} N_{i} \frac{\partial F_0}{\partial S_{i}} + 2\pi i \tau \sum_{i} S_{i} \]

where \( F_0 \) is a planar contribution of the free energy of the continuous matrix model.

Is there any direct connection between \( N=2 \) instanton partition function and \( N=1 \) glueball superpotential including all graviphoton corrections??
If we would like the instanton contribution of 5D theory, we need to take T-dual along $\phi$

$$\prod_{i<j} (g_s n_i - g_s n_j) \rightarrow \prod_{i<j} \prod_{m \in \mathbb{Z}} \left( g_s n_i - g_s n_j + i \frac{m}{R} \right)$$

$$= \prod_{i<j} \sinh(\pi g_s R (n_i - n_j))$$

Kakuza-Klein modes
Using the similar localization near the critical points, the partition function gives in the large N limit

$$Z_{5D} = \sum_{\vec{k}} \prod_{(l,i) \neq (n,j)} \left( \frac{\sinh(\pi R(a_l - a_n + g_s(k^{(l)}_i - k^{(n)}_j + j - i)))}{\sinh(\pi R(a_l - a_n + g_s(j - i)))} \right)^{1-G} \wedge^{2rk}$$

This trigonometric extension ($q$-deformation) also gives $q$-deformed 2D YM, 3D CS on $\Sigma_G \times S^1$, topological string amplitude on a CY$_3$, Gopakumar-Vafa invariant, etc.
Integrable subset of full string theory

- $c=0$ matrix model
- $N=1$ superpotential
- 4D $N=2$ instantons
- 2D Yang-Mills
- (IKKT like) D-instanton model
- Gross-Taylor / $c=1$ string
- Little string / Liouville theory
- 5D BPS particles
- 3D Chern-Simons
- Topological string on CY$_3$
- (BFSS like) D-particle model

- Critical string theory
- BFSS / IKKT
- M-theory
- F-theory
We derive the instanton (BPS state) counting from generalized 2D YM theory.

We found an essential structure of both partition function is "discrete matrix model".

T-dual to higher dimensional theory relates to many interesting physical / mathematical models.