A theoretical analysis of a hybrid meson

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2003 X(3872) observed at Belle

2004

2005 X(3915) at Belle - **Y(4260) at BaBar**

2006

2007 X(3940), Y(4008), Y(4660) at Belle –

2008 Z+-(4050), X(4160), Z+-(4250), Z+-(4430), X(4630) at Belle

2009 Y(4140) at CDF

2010 X(3915), X(4350), Yb(10888) at Belle

2011 Y(4274) at CDF

2012 Zb+-(10610) and Zb+-(10650) at Belle

2013 X(3823) & Zb0(10610) at Belle – Zc+-(3900) & Zc+-(4020) at BESIII

2014 Zc0(4020) at BESIII – Z+-(4200) at Belle – Z+-(4240) at LHCb

2015 Zc+-(4055) at Belle – Y(4230) at BESIII

Outline

a brief introduction of Y(4260)

   * Y(4260) & its interpretation
   * the selection rules

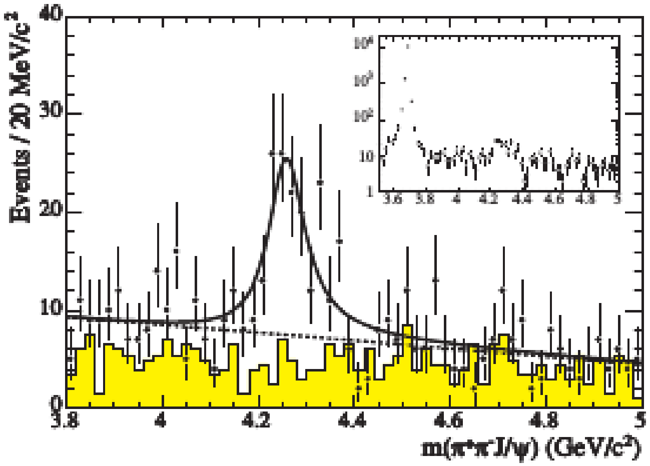
our approach
   * hyperspherical formalism
   * auxiliary field technique

the numerical results we obtained

summary

Y(4260)

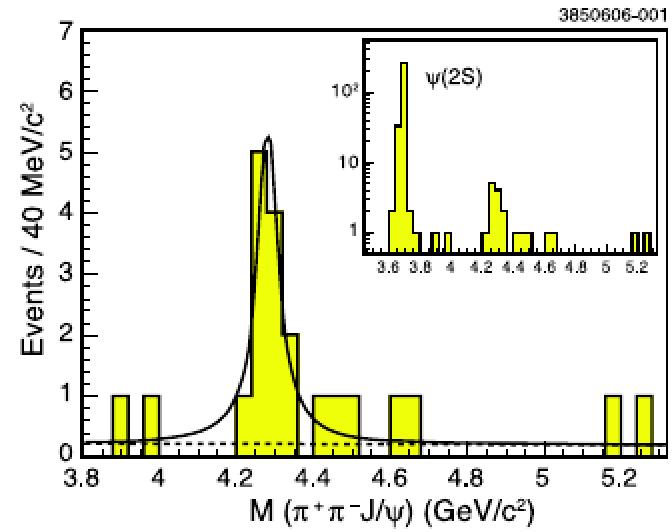
$$e^+e^- \to \gamma\pi^+\pi^- J/\psi$$

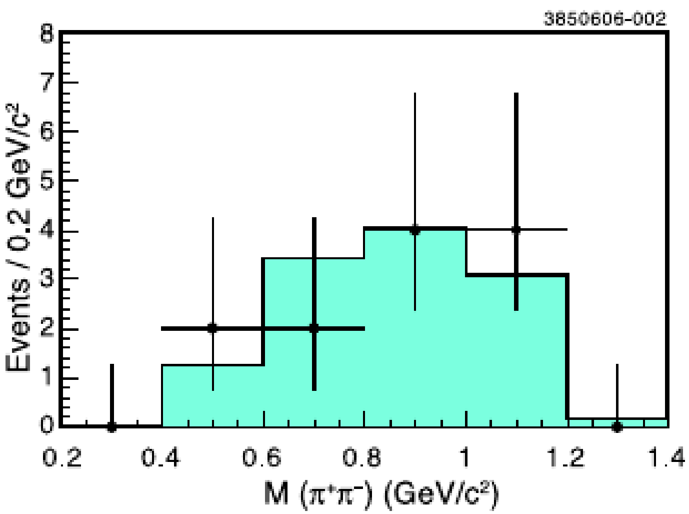


Discovered by the BaBar group through initial state radiation (ISR) events

Another evidence provided by the Cleo collaboration

Width ~ 90 MeV





Relativistic single resonance Breit-Wigner fit

The measured dipion mass distribution agreed with the theoretical Monte Carlo S-wave phase space model

$$\boxed{J^{PC} = 1^{--}}$$

B. Aubert et al, Phys. Rev. Lett 95 142001 (2005)
Q. He et al, Phys. Rev. D 74 091104 (2006)
J. Beringer et al, Phys. Rev. D 86 010001 (2012)

Y(4260)

Stat & Sys

The 2013 measurement: $$M(Y4260) = 4258.6 \pm 8.3 \pm 12.1$$

$$\Gamma_{\text{tot}} = 134.1 \pm 16.4 \pm 5.5$$

The 2016 measurement:
(K. Olive et al)

$$M(Y4260) = 4251 \pm 9$$

$$\Gamma = 120 \pm 12$$

A lattice calculation with a pion mass of about 400 MeV suggests there exists J^{PC}=1-- around 4280 MeV

L. Liu et al, Journal of High Energy Physics, 122 (2012)
Access to this journal is limited ARXIV: https://arxiv.org/pdf/1204.5425.pdf

Decays into $$J/\psi + \pi^-\pi^+$$

$$J/\psi + \pi^0\pi^0$$

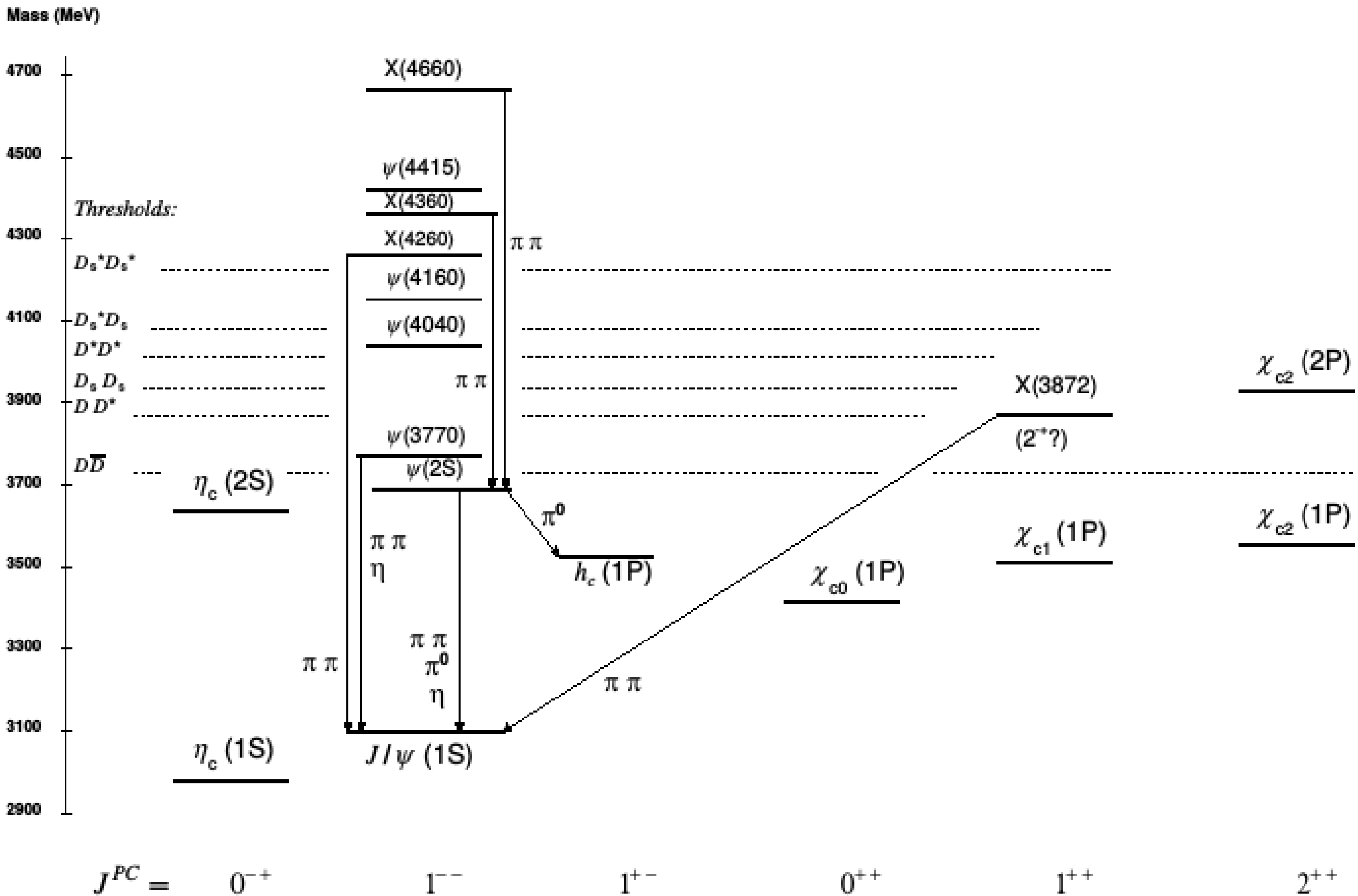
$$J/\psi + K^-K^+$$

Z(3900) $$M(Z3900) = 3894.5 \pm 6.6 \pm 4.5$$

$$\Gamma = 63 \pm 24 \pm 26 \quad \mathrm{MeV/c^2}$$

Z.Q. Liu et al, Phys. Rev. Lett. 110, 252002 (2013)

THE CHARMONIUM SYSTEM



Y(4260)

Tetraquark $c\bar{c}s\bar{s}$ interpretation
needs the channel of $D_s + \bar{D}_s$

DD_1 molecule interpretation

Y4260 close to DD_1

$$e^- e^+ \to \begin{cases} \pi^+\pi^- J/\psi \\ \pi^+\pi^- h_c \\ \omega\chi_{c0} \end{cases}$$

The single-resonance assumption is naïve to determine the mass & the width of Y(4260). → Average the mass and width determinations in the 3 channels - Y4260 label "retired"
(Olsen 2017)

$D\bar{D}_1$ B.E. soars to 66 MeV ⟵ $M(Y4220) = 4222 \pm 3$

$\Gamma_{\rm tot} = 48 \pm 7$

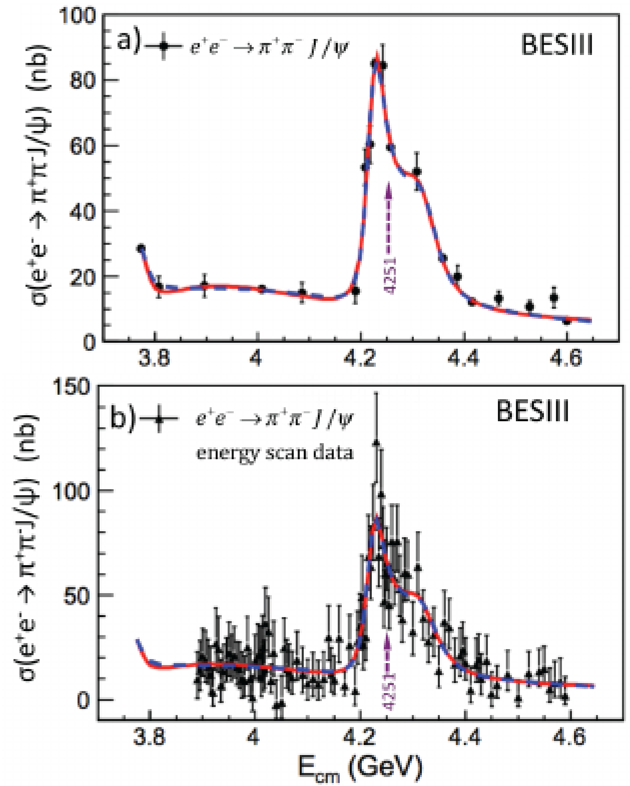
Hybrid meson

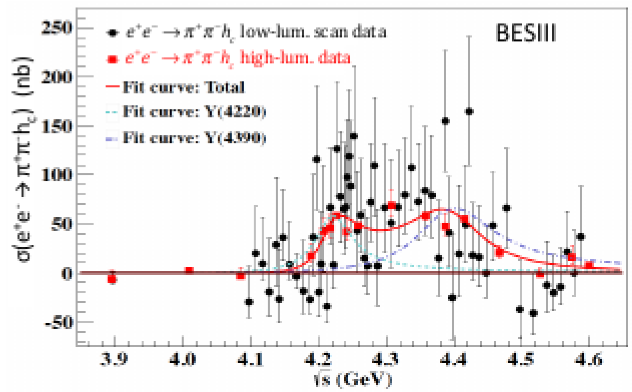
$H_B : c\bar{c} +$ P-wave gluon

Selection rule to restrict the decay of the hybrid
(The decay into two S-wave open charm mesons is prohibited)

Gui-Jun Ding, Phys. Rev. D79, 014001 (2009)
E. Kou & O Pene, Phys. Lett. B 631 (2005) 164

Cross section measurements

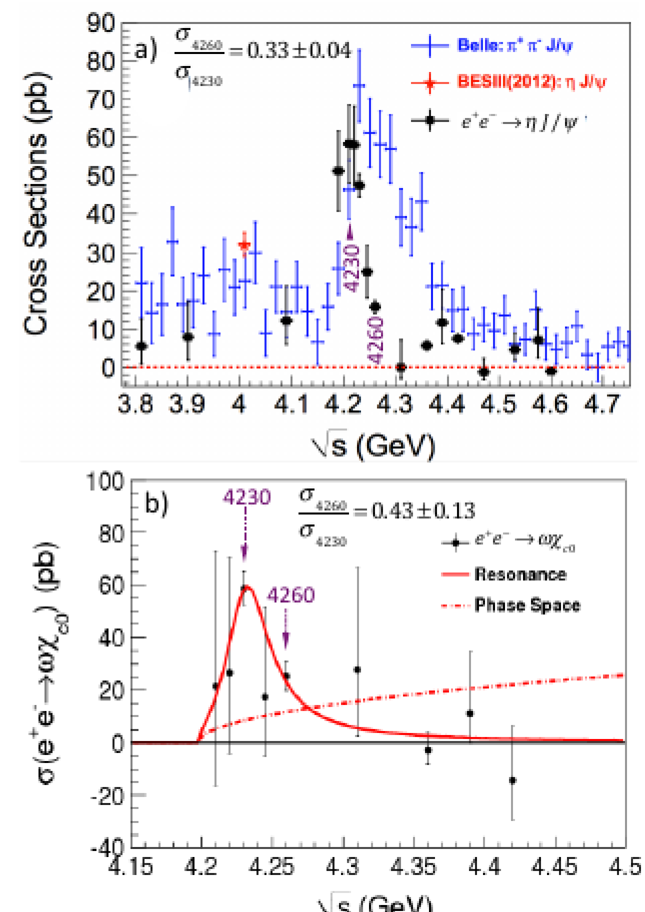




$M_1 = 4218 \pm 4$ MeV $\quad \Gamma_1 = 66 \pm 9$ MeV

$M_2 = 4392 \pm 6$ MeV $\quad \Gamma_2 = 140 \pm 16$ MeV

Y(4360) parameters inconsistent



Simplest interpretation:
The first peak ← Y(4260)
The second ← Y(4360)

$M_1 = 4222 \pm 4$ MeV $\quad \Gamma_1 = 44 \pm 5$ MeV

$M_2 = 4320 \pm 13$ MeV $\quad \Gamma_2 = 101^{+27}_{-22}$ MeV,

Hybrid meson

A bound system which consists of a quark, antiquark and gluon

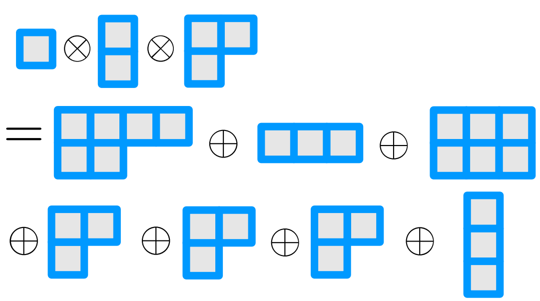
Quarks heavy and slow $\Rightarrow$ NR $\mathcal{O}(m_q^{-1})$

Interaction between a quark and gluon is an attractive linear potential

Quark-antiquark interaction weak & repulsive
If it is exotic (Beringer 2012), then

$$J^{\mathrm{PC}} = 0^{\pm -}, 1^{-+}, 2^{+-}, \ldots$$

Gluon carries the adjoint representation **8** of SU(3) – it can be linked to a quark & antiquark to form a colour singlet object.



$$3 \otimes \bar{3} \otimes 8 = 27 \oplus 10 \oplus \overline{10} \oplus 8 \oplus 8 \oplus 8 \oplus 1$$

D. Horn & J. Mandula, Phys. Rev. D 17 (1978) 898

Hybrid meson

Hybrid charmonium as a bound state of $c\bar{c}$ plus a gluon

For a magnetic (transverse electric) gluon:

$$L_g = J_g$$

$$P = (-1)^{(L_{q\bar{q}} + J_g)} \ , \quad C = (-1)^{L_{q\bar{q}} + S_{q\bar{q}} + 1}$$

The lowest state is: $L_{q\bar{q}} = 0 \ , \quad J^{PC} = 1^{--}$

For an electric (transverse magnetic) gluon:

$$L_g = J_g \pm 1$$

$$P = (-1)^{(L_{q\bar{q}} + J_g + 1)} \ , \quad C = (-1)^{L_{q\bar{q}} + S_{q\bar{q}} + 1}$$

A. Le Yaouanc et al, Z. Phys. C – Particle & Physics 28, 309 (1985)

Note:

Cf: electric,magnetic photon (radiation) carries parity of: $(-1)^l, (-1)^{l+1}$

The parity of a meson is:

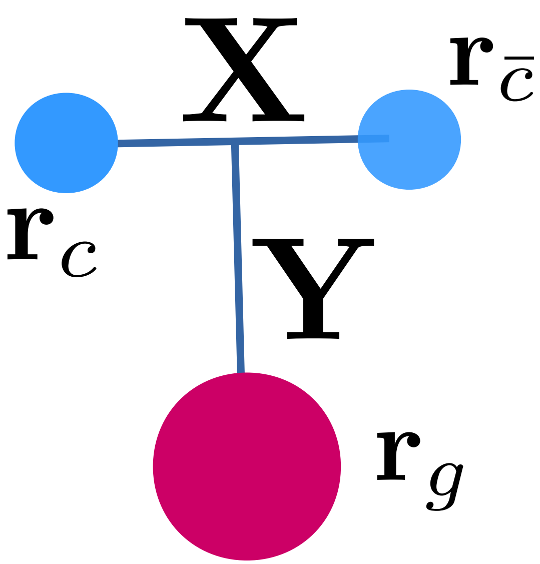
$$P = (-1)^{L+1}$$

From the selection rules that we abide by

Table 1: A hybrid meson's states which are allowed to exist.

States which are allowed to exist for a hybrid meson $q\bar{q}g$									
Gluon type	$L_{q\bar{q}}$	L_g	L_{tot}	$S_{q\bar{q}}$	S_g	S_{tot}	$J_{q\bar{q}}$	J_g	J^{PC}
E	1	0	1	1	1	0,1,2	0,1,2	1	1^{--}
E	0	1	1	0	1	1	0	0	1^{--} forbidden
E	2	1	1,2	2	1	1,2,3	1	0	1^{--}
M	0	1	1	0	1	1	0	1	1^{--}
M	2	1	1,2,3	2	1	1,2,3	0,1,2	1	1^{--}
E	1	2	1,2,3	1	1	0,1,2	0,1,2	1	1^{--}
E	3	0	3	3	1	2,3,4	0,1,2	1	1^{--}

Hyperspherical formalism



$$\mathcal{Y}^{\mathrm{L_x L_y}}_{\mathrm{KLM_L}}(\Omega_5) \stackrel{\mathrm{def}}{=} \mathcal{N}(K, L_x, L_y) A_n^{L_x L_y}(\alpha) \left[Y_{L_x}(\Omega_x) \otimes Y_{L_y}(\Omega_y) \right]_{LM_L}$$

$$A_n^{L_x L_y}(\alpha) = (\cos\alpha)^{L_x} (\sin\alpha)^{L_y} P_n^{L_y+1/2, L_x+1/2}(\cos 2\alpha)$$

$$\mathcal{N}(K, L_x, L_y) = \sqrt{2(K+2)} \sqrt{\frac{\Gamma(n+1)\Gamma(L_x+L_y+n+2)}{\Gamma(L_x+n+3/2)\Gamma(L_y+n+3/2)}}$$

$$\Psi_{JM_J}(\mathbf{x}, \mathbf{y}) = \frac{1}{\rho^{5/2}} \sum_{K,\gamma} \chi_{K,\gamma}(\rho) \Upsilon^{JM_J}_{K\gamma}(\Omega_5)$$

$$\Upsilon^{JM_J}_{K\gamma}(\Omega_5) = \left[\mathcal{Y}^{\mathrm{L_x L_y}}_{\mathrm{KL}} \otimes |S\rangle \right]_{JM_J}$$

$$\left[\frac{\mathbf{p}_x^2}{2m_x} + \frac{\mathbf{p}_y^2}{2m_y} + (\ldots) \right] \Psi = E\Psi$$

$$-\frac{\hbar^2}{2m_b} \left[\frac{\partial}{\partial \rho^2} - \frac{(K+3/2)(K+5/2)}{\rho^2} + (\ldots) \right] \chi = E\chi$$

A. Novoselsky et al. Phys. Rev. A 49, 833 (1994)

The Hamiltonian for the Y4260

$$H(\mu, \mu_g) = H_0 + V$$

$$H_0 = \mu + \frac{\mu_g}{2} + \frac{m^2}{\mu} + \frac{\mathbf{p}_X^2}{\mu} + \frac{\mathbf{p}_Y^2}{2\phi}$$

$$V = \sigma \left| r_{qg} \right| + \sigma \left| r_{\bar{q}g} \right| + V_{\mathrm{C}}$$

$$V_C = -\frac{3\alpha_s}{2r_{qg}} - \frac{3\alpha_s}{2r_{\bar{q}g}} + \frac{\alpha_s}{6r_{q\bar{q}}}$$

σ : energy density

α_s : the strong coupling constant

$$\mathbf{X} = \mathbf{r}_q - \mathbf{r}_{\bar{q}}$$

$$m_q = m_{\bar{q}} = m$$

$$\mathbf{Y} = \mathbf{r}_g - \frac{\mathbf{r}_q + \mathbf{r}_{\bar{q}}}{2}$$

$$\mu_q = \mu_{\bar{q}} = \mu$$

V. Mathieu, Phys. Rev. D **80**, 014016 (2009)

The einbein field formalism crucial to the QCD string model

$$\sqrt{\mathbf{p}^2 + m^2} + V(r) \Longleftrightarrow \frac{\mathbf{p}^2 + m^2}{2\mu} + \frac{\mu}{2} + V(r)$$

μ: einbein field

More complicated numerical calculation. Another disadvantage of this form is that the action cannot be used to describe massless particles.

Thus need to re-parametrise this with auxiliary fields

Its form is non-relativistic, while its dynamics is relativistic. (In addition, its quantisation becomes easier in a path integral framework. In a string language, the Nambu-Goto action's form is too awkward to quantise.)

We have an additional degree of freedom.

$$S = \sqrt{-m^2} \int d\tau \sqrt{\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} \quad \overset{\text{By EoM}}{\Longleftrightarrow} \quad S = \frac{1}{2} \int d\tau \left(\frac{\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}}{\mu} - \mu m^2 \right)$$

cf.

K. Becker et al. String theory and M-theory: A modern introduction (CUP) 2007

Yu. S. Kalashnikova & A. V. Nefediev, Physics of Atomic Nuclei, Vol. 68, No. 4, 2005, pp 650-660

Parameters

Fitted to the lowest two states of the charmonium

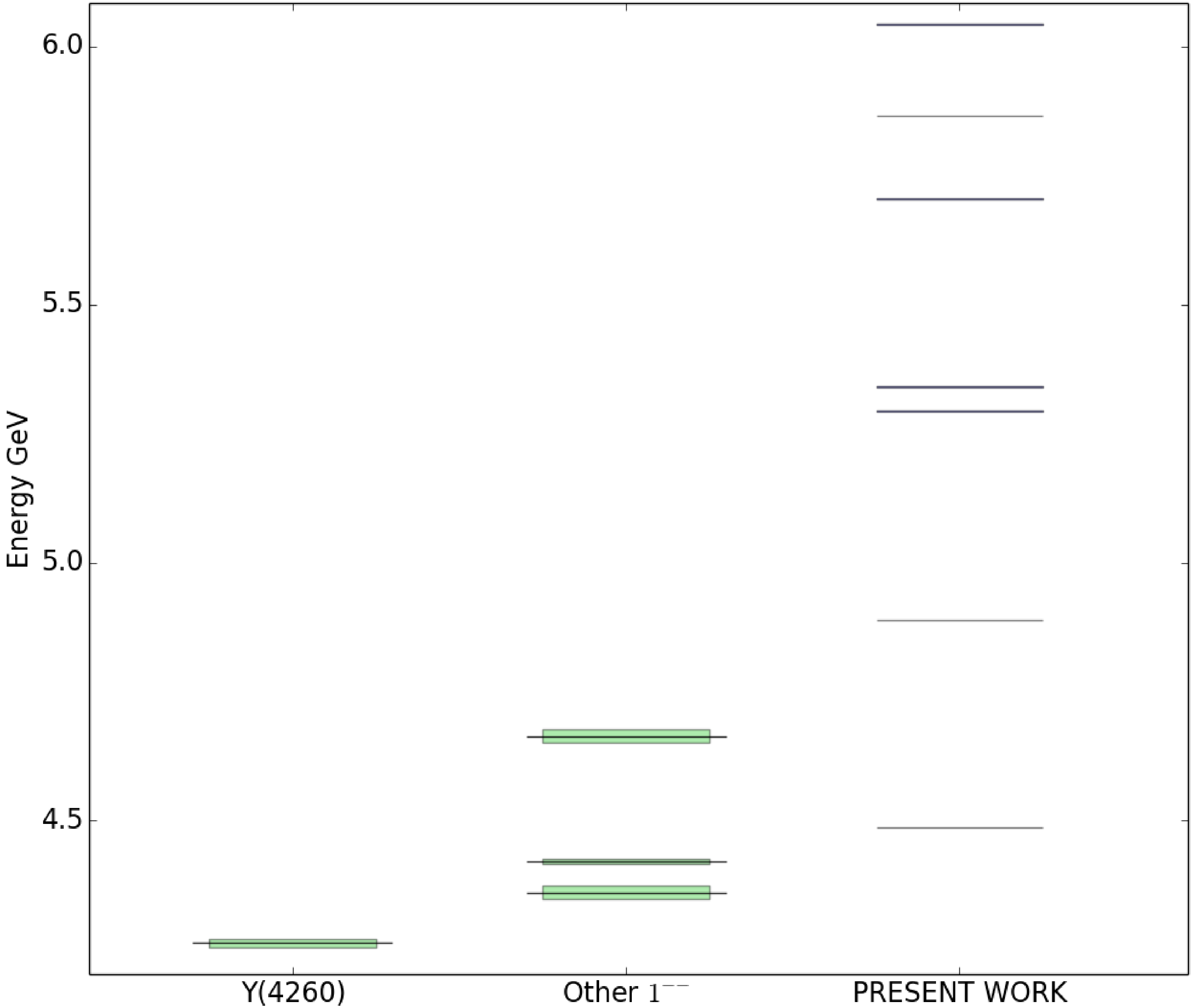
$$^1S_0 \qquad ^3S_1 \quad \Rightarrow \quad \begin{aligned} \sigma &= 0.16\ \mathrm{GeV}^2 \\ \alpha_s &= 0.55 \\ m_c &= 1.48\ \mathrm{GeV} \end{aligned}$$

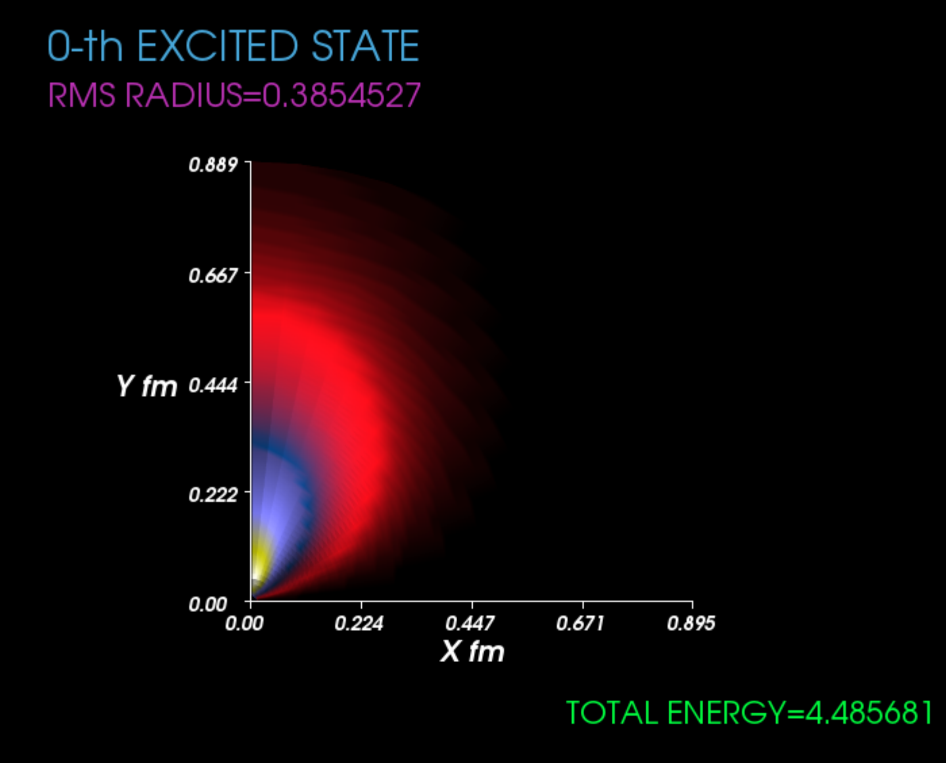
Auxiliary field technique

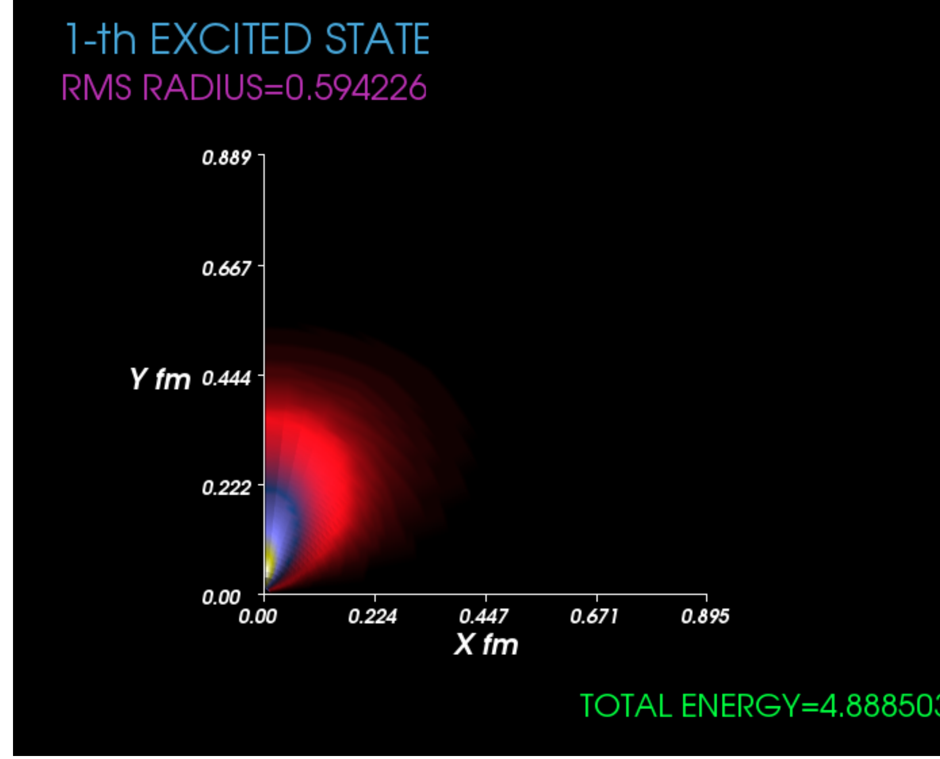
$$\left.\frac{\delta H}{\delta \mu_c}\right|_{\mu_c=\mu_{c0}} = 0$$

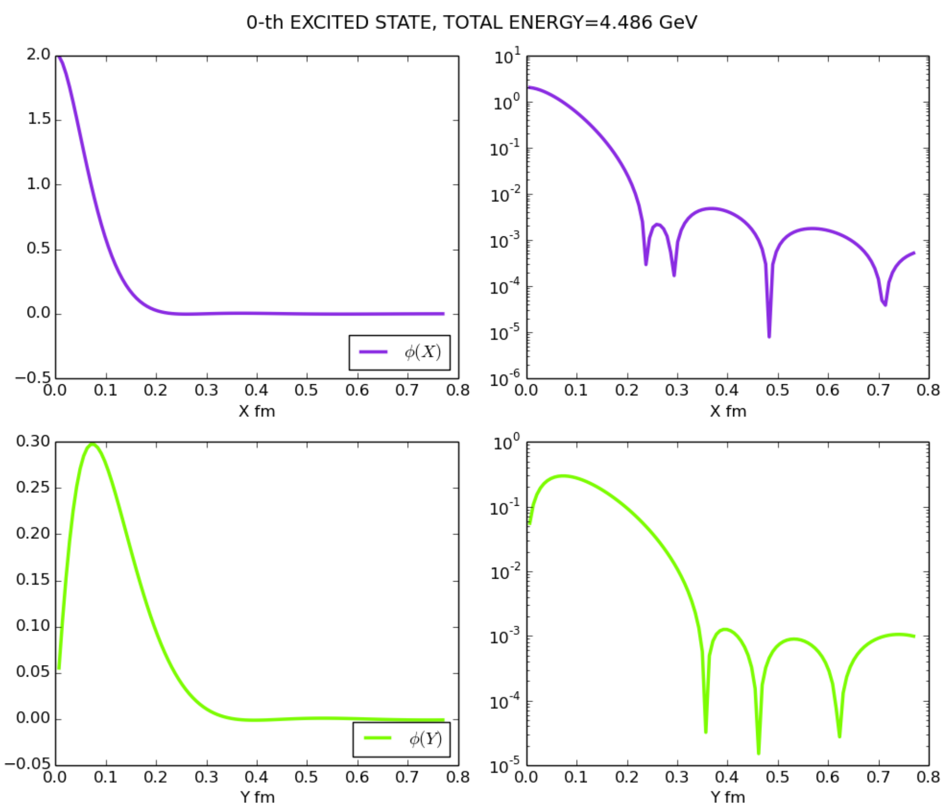
$$\left.\frac{\delta H}{\delta \mu_g}\right|_{\mu_g=\mu_{g0}} = 0 \qquad\qquad \mu_c = 1.598\ , \quad \mu_g = 1.085$$

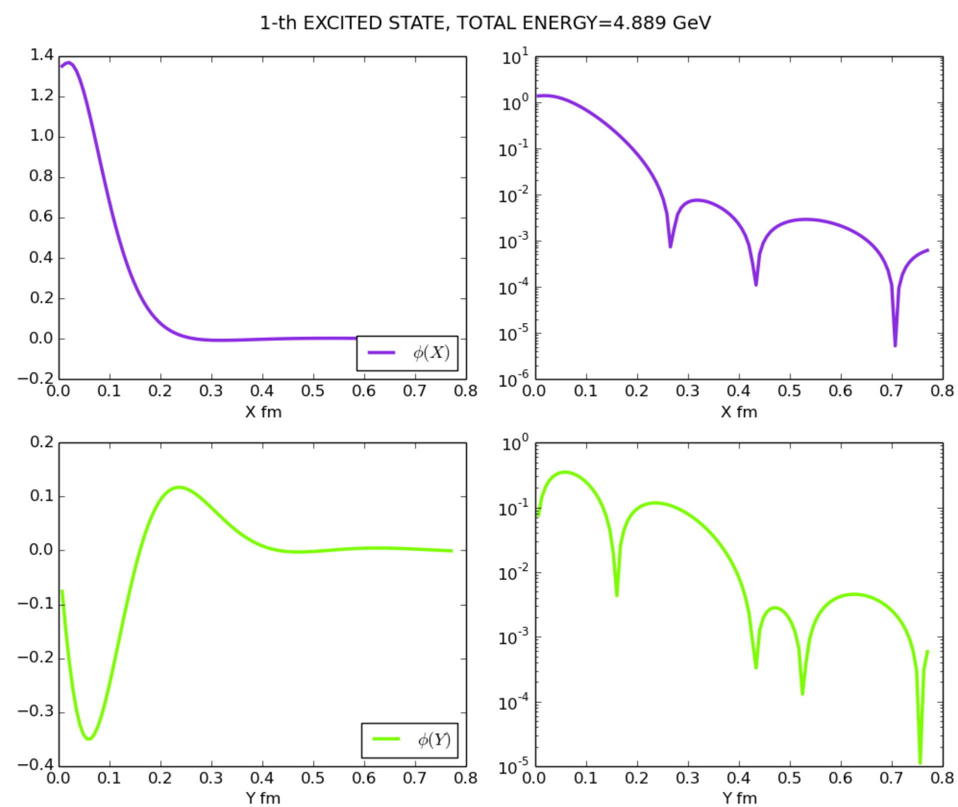
Y(4260) spectrum




0-th EXCITED STATE
RMS RADIUS=0.3854527
Y fm
0.889
0.667
0.444
0.222
0.00
0.00
0.224
0.447
0.671
0.895
X fm
TOTAL ENERGY=4.485681


1-th EXCITED STATE
RMS RADIUS=0.594226
Y fm
0.889
0.667
0.444
0.222
0.00
0.00
0.224
0.447
0.671
0.895
X fm
TOTAL ENERGY=4.888503


0-th EXCITED STATE, TOTAL ENERGY=4.486 GeV
$\phi(X)$
X fm
$\phi(Y)$
Y fm


1-th EXCITED STATE, TOTAL ENERGY=4.889 GeV
$\phi(X)$
X fm
$\phi(Y)$
Y fm

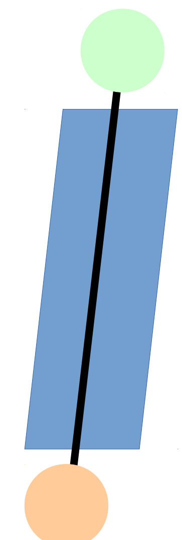
Quark-antiquark effective potential

Characterised by:

Λ Projection of the total angular momentum of a gluon onto the $q\bar{q}$ axis

(+/-) Reflection in the plane which contains the axis

(g/u) charge conjugation & the spatial inversion of $q\bar{q}$



The static quark potential cannot be directly measured in an experiment

The hadronic scale parameter defined by the interaction between static quarks

$$r^2 \frac{dV(\mathbf{r})}{dr}\bigg|_{r=r_0} = 1.65$$

Phenomenological potential models

$$r_0 \approx 0.5 \text{ fm}$$

$V(\mathbf{r})$: The static quark potential

C. Morningstar & M. Peardon, Phys. Rev. D 56 4043 (1997)
R. Sommer, Nucl. Phys. B411, 839 (1994)

Quark-antiquark effective potential

Modification to the Cornel potential → the Luscher term

$$V_{q\bar{q}} = ar + \frac{\pi}{r}\left(N - \frac{1}{12}\right)$$

$$V_{q\bar{q}} = \sqrt{a^2r^2 + 2\pi aN} + \frac{\alpha_s}{6r}$$

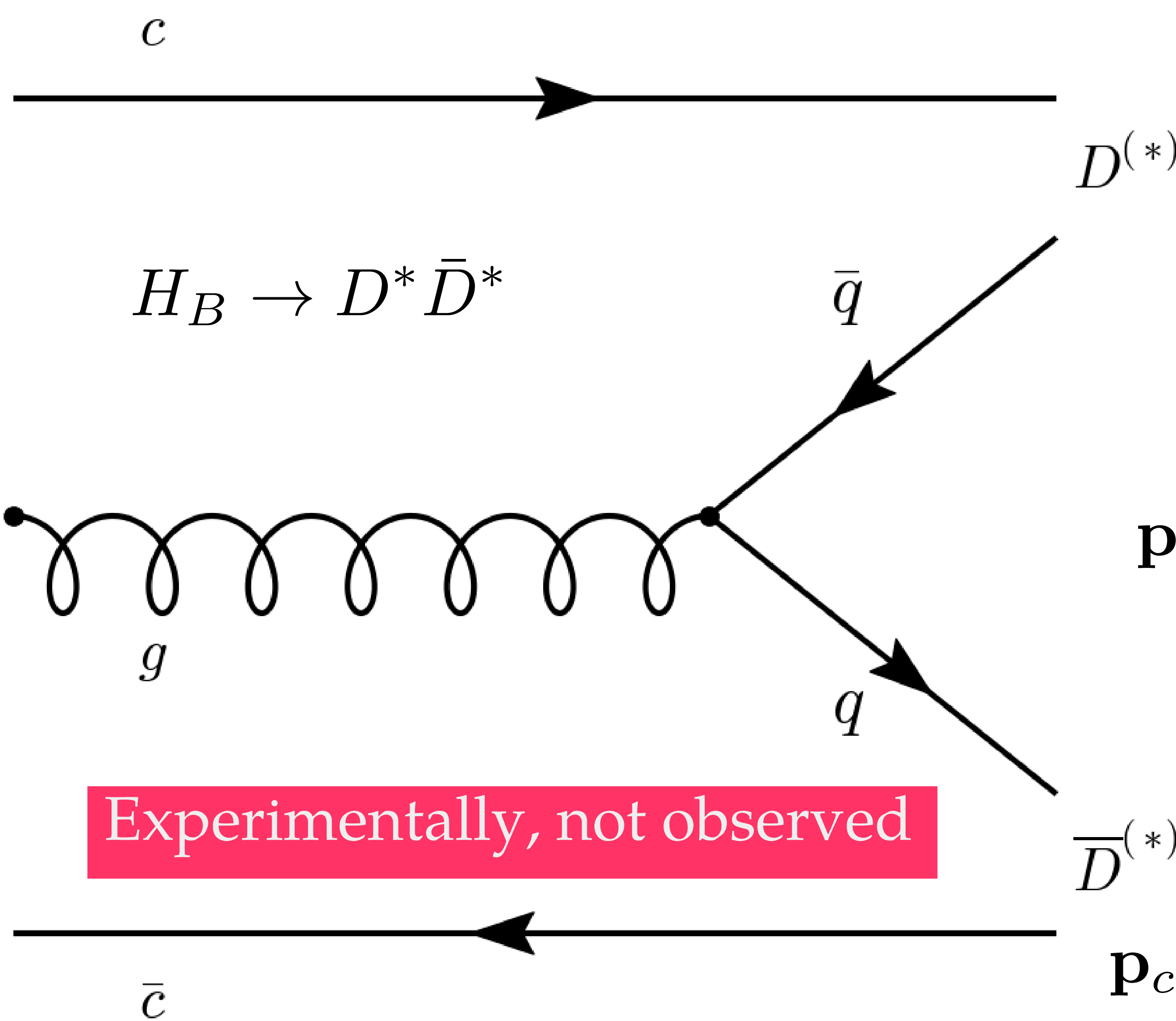
$\boxed{q\bar{q} \ \mathbf{8}}$

$$\downarrow$$

$$\approx ar + \frac{\pi N}{r}$$ At large distances

N: The excitation number of string

F. Buisseret at al, Eur. Phys. J. A 29, 343-351 (2006)



$H_B \to D^* \bar{D}^*$

Experimentally, not observed

$A \to BC$

$$\mathbf{p}_{q\bar{q}} = \frac{\mathbf{p}_q - \mathbf{p}_{\bar{q}}}{2}$$

$$\mathbf{p}_B = \frac{m_q \mathbf{p}_c - m_c \mathbf{p}_{\bar{q}}}{m_c + m_q}$$

$$\mathbf{p}_{cc} = \mathbf{p}_{q\bar{q}} - \mathbf{p}_f$$

$$\mathbf{p}_B = -\frac{m_q \mathbf{p}_f}{m_q + m_c} + \mathbf{p}_{q\bar{q}} - 0.5\mathbf{k}$$

$$\mathbf{p}_C = \frac{m_q \mathbf{p}_f}{m_q + m_c} - \mathbf{p}_{q\bar{q}} - 0.5\mathbf{k}$$

From the selection rules that we abide by

$$I = \int \frac{d\mathbf{p}_{cc} d\mathbf{k}}{\sqrt{2\omega}(2\pi)^6} \Psi_{l_B}^{m_B *}(\mathbf{p}_B) \Psi_{l_C}^{m_C *}(\mathbf{p}_C) \Psi_{l_{H_B}}^{m_{H_B}}(\mathbf{p}_{c\bar{c}}, \mathbf{k}) d\Omega_f Y_l^{m*}(\Omega_f)$$

$\pm\mathbf{p}_f$:momentum of the final mesons

$$\mathbf{p}_c + \mathbf{p}_{\bar{c}} = -\mathbf{p}_{\bar{q}} - \mathbf{p}_q = -\mathbf{p}_g$$

$$\mathbf{p}_{\bar{c}} + \mathbf{p}_q = -\mathbf{p}_{\bar{q}} - \mathbf{p}_c \equiv \mathbf{p}_f$$

I: odd function with regard to $\mathbf{k}$

The hybrid WF is odd for $\mathbf{k}$ as $l_g = 1$

S-wave mesons' WFs identical

Only S-wave-gluon hybrid charmonium can decay into DD

Decay widths of a hybrid meson psi4160

$J_{q\bar{q}}$	$\Gamma_{D^0\bar{D}^0}$	$\Gamma_{D^+D^-}$	$\Gamma_{D^{*0}\bar{D}^0}$	$\Gamma_{D_s^+D_s^-}$	$\Gamma_{D^{*0}\bar{D}^{*0}}$			
					$S=0$	$S=1$	$S=2$	
0	0.1328	0.1366	0.3702	0.1039	0.0586	0	1.1727	
1	0.3986	0.4099	0.2777	0.3117	0.1759	0	0.8795	
2	0.6644	0.6832	0.4628	0.5196	0.2931	0	0.0586	

$$\Gamma = 0.103 \pm 0.008$$

Beringer 2012

$$\psi(4160) : J^{PC} = 1^{--}$$

$$m_s = 0.5 \ \ \mathrm{GeV}$$

kmax=pmax=2.6 $(p_r, p_\theta, p_\phi, \theta_B, \phi_B) = (55, 10, 10, 10, 10)$

Summary

i. Y(4260), discovered more than a decade ago, is a hybrid meson candidate

ii. We have carried out an indepth analysis of the particle by adopting hyperspherical formalism & auxiliary field technique

iii.→ Spectrum above the experimental data, but some additional factors (channel coupling etc) may make our theory more consistent with the experimental data

iv.Quark-antiquark effective potentials extracted – it was below the lattice calculation. Suggesting the single gluon assumption was naive.

v.Decay width of psi4160 (1--) too large → likewise that of Y(4260) may be

The modification of the selection rule
- Mixing with states close to Y4260, eg psi(4160). But small effects due to small overlap stemming from many nodes of the wave functions (second order mechanism)

- Lorentz covariant effect on the light quarks (difficult to estimate)

- Channel coupling effects

Other modes
Not forbidden

$$Y(4260) \to D^{**}\bar{D}^{*} \to D^{*}\bar{D}^{*}\pi' s$$

$$Y(4260) \to D^{*}\bar{D}^{**} \to D^{*}\bar{D}^{*}\pi' s$$

$Y(4260)$ sits below the $D^{**}\bar{D}^{*}$ thresholds.

Resonance not narrow.

Possibility of dominant $Y(4260) \to D^{*}\bar{D}^{*}\pi' s$

Probability densities

Integrated with regard to the angular variables of **x** and **y**: the resulting function is a function of radial components of them.

$$g_C(X,Y) = \int |\Psi|^2 d\Omega_x d\Omega_y$$

$$= \frac{1}{\rho^5} \sum_{K\gamma,K'} \chi^*_{K',\gamma}(\rho)\chi_{K\gamma}(\rho) f^{K'L_yL_x}_{KL_yL_x}(\alpha)$$

$$f^{K'L_yL_x}_{KL_xL_y}(\alpha) = \mathcal{N}(K',L_y,L_x) P^{L_y+1/2,L_x+1/2}_{n'}(\cos 2\alpha)$$
$$\times \mathcal{N}(K,L_y,L_x) P^{L_y+1/2,L_x+1/2}_{n}(\cos 2\alpha)$$
$$\times (\cos\alpha)^{2L_x}(\sin\alpha)^{2L_y}$$

$$g_M(k_x,k_y) = \int |\Phi_{JM_J}|^2 \frac{d\Omega_{kx}}{(2\pi)^2}\frac{d\Omega_{ky}}{(2\pi)^2}$$

$$= (2\pi)^2 \sum_{K\gamma K'} \sum_{K_c K'_c} U_{K_c\gamma K'_c} U_{K\gamma K'} (-i)^{K'_c} i^{K'} f^{K'_c L_x L_y}_{K' L_x L_y}(\alpha_k)$$

Single-variable wave functions

$\phi(X),\ \phi(Y)$

$$\begin{aligned}\Psi_{\rm tot} &= \frac{u(X,Y)}{XY}\left[Y_{L_x}(\Omega_X)\otimes Y_{L_y}(\Omega_Y)\right]_L \\ &= \frac{1}{\rho^{5/2}}\sum_{K,\gamma}\chi_{K\gamma}\mathcal{N}A_n^{L_xL_y}(\alpha)\left[Y_{L_x}(\Omega_X)\otimes Y_{L_y}(\Omega_Y)\right]_L\end{aligned}$$

RMS hyper-radius & radius

$$\langle \rho^2 \rangle = \int |\Psi_{JM_J}|\rho^2 d\mathbf{x}d\mathbf{y} = \sum_{K,\gamma,M_L} \int_0^\infty |\chi_{K\gamma}|^2 \rho^2 d\rho$$

$$\langle r^2 \rangle = \frac{1}{3}\langle r_q^2 + r_{\bar{q}}^2 + r_g^2 \rangle$$ Regarding the 3 particles as point particles

If the mass of particle 1 and that of particle 2 are the same, the RMS radius are more easily calculated:

$$\mathbf{r}_1^2 + \mathbf{r}_2^2 + \mathbf{r}_3^2 = \left(\frac{m_1^2 + m_2^2 + m_3^2}{M m_1 m_3} (\sin\alpha)^2 + \frac{1}{m_1} (\cos\alpha)^2 \right) \rho^2 (\hbar c)^2$$

Then we calculate $\sqrt{\langle \rho^2 \rangle}, \quad \sqrt{\langle r^2 \rangle}$

cf. D.V. Fedorov et al, Phys.Lett. B 389 (1996), 631-636
B.V. Danilin et al, Phys. Rev. C 71, 057301 (2005)

Charmonium spectrum calculated within the QCD string framework

$$H_{\rm tot}^{c\bar c} = H_0^{c\bar c} + V_{\rm Lin+Cou}^{c\bar c} + V_{\rm str}^{c\bar c} + V_{\rm LS}^{c\bar c} + V_{\rm ss}^{c\bar c} + V_{ST}^{c\bar c}$$

$$H_0^{c\bar c} + V_{\rm Lin+Cou}^{c\bar c} = 2\sqrt{\mathbf{p}^2 + m^2} + \frac{\sigma r}{\hbar c} - \frac{4}{3}\frac{\alpha_s \hbar c}{r}$$

$$\to \frac{m^2}{\mu} + \mu + \frac{\mathbf{p}^2}{\mu} + \frac{\sigma r}{\hbar c} - \frac{4}{3}\frac{\alpha_s \hbar c}{r}$$

$$V_{\rm str}^{c\bar c} = -\frac{\sigma \mathbf{L}^2}{6\mu^2 r}(\hbar c)$$

$$V_{\rm LS}^{c\bar c} = -\frac{\sigma}{2\mu^2 r}(\mathbf{L}\cdot\mathbf{S})(\hbar c) + \frac{2\alpha_s}{\mu^2 r^3}(\mathbf{L}\cdot\mathbf{S})(\hbar c)^3$$

$$V_{\rm ss}^{c\bar c} = \frac{32\pi\alpha_s}{9\mu^2}(\mathbf{s}_q\cdot\mathbf{s}_{\bar q})\delta(\mathbf{r})(\hbar c)^3$$

$$V_{\rm ST}^{c\bar c} = \frac{4\alpha_s}{3\mu^2 r^5}\left[3(\mathbf{s}_q\cdot\mathbf{r})(\mathbf{s}_{\bar q}\cdot\mathbf{r}) - r^2(\mathbf{s}_q\cdot\mathbf{s}_{\bar q})\right](\hbar c)^3$$

$$\sigma = 0.16\ \mathrm{GeV}^2$$

$$\alpha_s = 0.55$$

$$m_c = 1.48\ \mathrm{GeV}$$

Yu. S. Kalashnikova et al., Phys. Rev. D 64, 014037 (2001)

Yu. S. Kalashnikova & A. V. Nefediev, Phys. Rev. D **77**, 054025 (2008)

The spin-spin interaction

Smearing technique to deal with the delta function issue

$$\delta(\mathbf{r}) = \frac{\delta(r)}{2\pi r^2}$$

The three-dimensional delta function in the spherical coordinates

$$\delta(r) \to \frac{\Lambda^2}{4\pi r} e^{-\Lambda r}$$

$$\Lambda = 3.5 \quad 1/\mathrm{fm}$$

Smearing

The spin-spin operator is dealt with by the following identity:

$$\mathbf{s}_1 \cdot \mathbf{s}_2 = \frac{1}{2}\left(S(S+1) - \frac{3}{4} - \frac{3}{4}\right)$$

T. Yoshida et al., Phys. Rev. D 92, 114029 (2015)

The tensor-type interaction

$$
\begin{aligned}
V_{\mathrm{ST}}^{c\bar{c}} &= \frac{4\alpha_s}{3\mu^2 r^5}\left[3(\mathbf{s}_q \cdot \mathbf{r})(\mathbf{s}_{\bar{q}} \cdot \mathbf{r}) - r^2(\mathbf{s}_q \cdot \mathbf{s}_{\bar{q}})\right] \\
&= \frac{\alpha_s}{3\mu^2 r^3} S_{12} \\
&\to \frac{\alpha_s (1 - e^{-\Lambda r})^2}{3\mu^2 r^3} S_{12}
\end{aligned}
$$

modification

$$
\begin{aligned}
S_{12} &= 12(\mathbf{s}_q \cdot \mathbf{n})(\mathbf{s}_{\bar{q}} \cdot \mathbf{n}) - 4(\mathbf{s}_q \cdot \mathbf{s}_{\bar{q}}) \\
&= 6(\mathbf{S} \cdot \mathbf{n})^2 - 2\mathbf{S}^2
\end{aligned}
$$

Spin singlet states are not affected by the tensor-type force

Yu. S. Kalashnikova et al., Phys. Rev. D 64, 014037 (2001)
T. Yoshida et al., Phys. Rev. D 92, 114029 (2015)

The tensor-type interaction

Eigenstate denoted by:

$$|LSJM_J\rangle$$

$$S_{12}|J+01JM_J\rangle = 2|J1JM_J\rangle$$

$$S_{12}|J-11JM_J\rangle = \frac{-2(J-1)}{2J+1}|J-11JM_J\rangle + \frac{6\sqrt{J(J+1)}}{2J+1}|J+11JM_J\rangle$$

$$S_{12}|J+11JM_J\rangle = \frac{-2(J+2)}{2J+1}|J+11JM_J\rangle + \frac{6\sqrt{J(J+1)}}{2J+1}|J-11JM_J\rangle$$

N. F. Mott & H. S. Massey, The Theory of Atomic Collisions, Oxford University Press, third ed. (1965)

The tensor-type interaction

$$S_{12}|110M_J\rangle = -4|110M_J\rangle$$

$$S_{12}|111M_J\rangle = 2|111M_J\rangle$$

No mixing of the S and D states

$$\psi = \psi_S + \cancel{\psi_D}$$

$$S_{12}|011M_J\rangle = 0$$

No mixing of the P and F states

$$\psi' = \psi'_P + \cancel{\psi'_F}$$

$$S_{12}|112M_J\rangle = -\frac{2}{5}|112M_J\rangle$$

The calculated spectrum of charmonium

Generalised Laguerre expansion method was used

	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_1$	$^3P_0$	$^3P_2$
Exp	2.981	3.096	3.525 hc	3.510	3.414	3.556
Kalashnikova	2.981	3.104	3.528	3.514	3.449	3.552
This work GS	3.036	3.072	3.537	3.509	3.424	3.578
Exp	3.638 etac	3.686 psi(2S)				3.927 chic2(2P)
First excited state	3.702	3.719	3.992	3.958	3.901	4.039

Mu=1.720 GeV, Lambda=3.5 1/fm
GL-alpha=2

The calculated spectrum of charmonium

Generalised Laguerre expansion method was used

	${}^1S_0$	${}^3S_1$	${}^1P_1$	${}^3P_1$	${}^3P_0$	${}^3P_2$
Exp	2.981	3.096	3.525 hc	3.510	3.414	3.556
Kalashnikova	2.981	3.104	3.528	3.514	3.449	3.552
This work GS	3.005	3.079	3.536	3.509	3.408	3.580

Exp	3.638 etac	3.686 psi(2S)				3.927 chic2(2P)
First excited state	3.686	3.720	3.996	3.967	3.909	4.046

Mu=1.720 GeV, Lambda=6.0 1/fm
GL-alpha=2, 30 basis functions