Parton Distribution Functions on the Lattice

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『ハドロン・原子核物理の理論研究最前線　２０１７』

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Plan

- Introduction for PDFs
  - Factorization
  - Global QCD analysis
- Quasi-PDFs
- Renormalization of non-local operator
  - Power divergence subtraction
  - Nonperturbative renormalization
- Quasi-PDFs: some results
- Pseudo-PDFs
- Summary and outlook

Study of nucleon structure

- Our mission

  - Understanding the inside the nucleon
  - Unraveling how quarks+gluons+interaction make up the nucleon

- Experimental tools

  - High-energy scattering at LHC, EIC, RHIC, ….

- Theoretical tools

  - Quantum Chromodynamics (QCD)
  - Perturbative QCD
  - Lattice QCD
  - Model

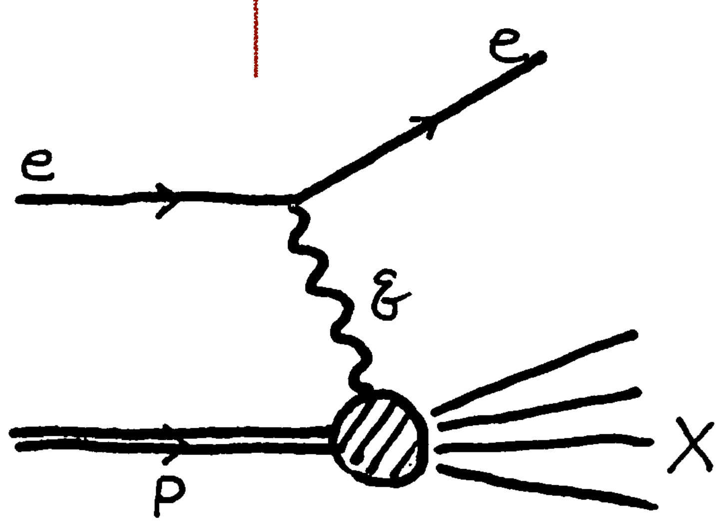
Factorization and PDFs

- Factorization theorem - a key concept in PQCD

$$\sigma^{\mathrm{DIS}}(x, Q^2, \sqrt{s}) = \sum_{\alpha=q,\bar{q},g} C_\alpha\left(x, \frac{Q^2}{\mu^2}, \sqrt{s}\right) \otimes f_\alpha(x, \mu^2) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right)$$

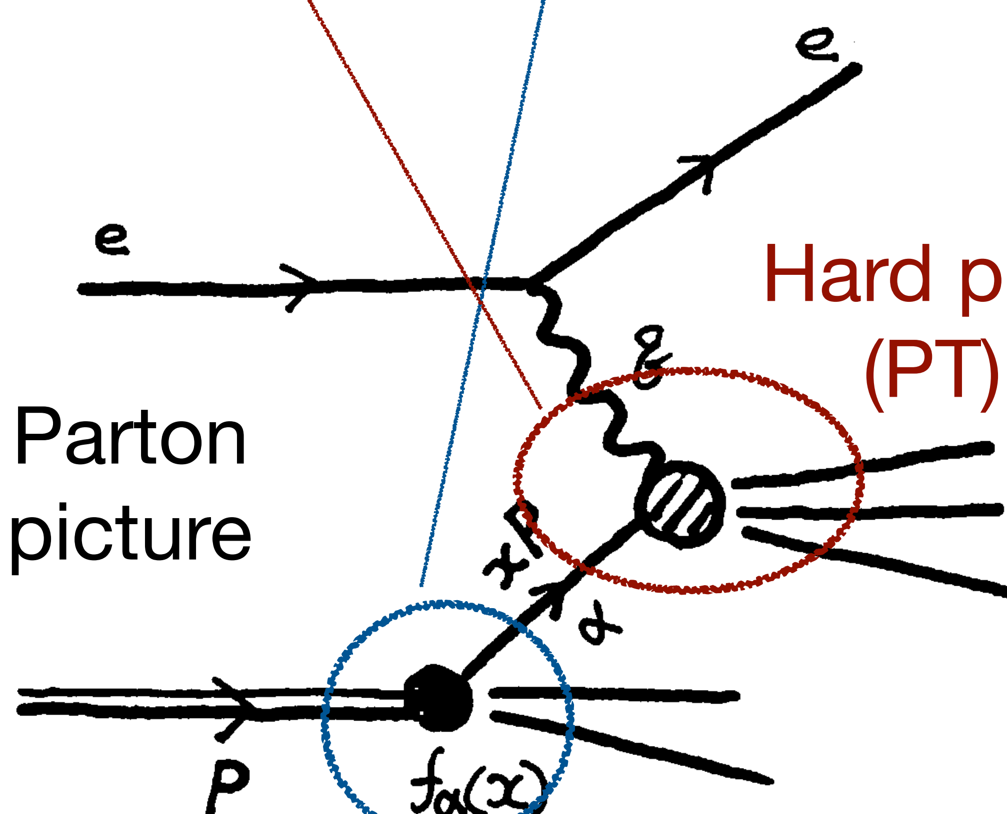
x : Bjorken-x, Q : momentum transfer, $\sqrt{s}$: collision energy

μ : factorization scale



Deep Inelastic Scattering (DIS)

Parton picture



Hard part (PT)

PDFs (NPT)

Factorization and PDFs

- Factorization theorem - a key concept in PQCD

$$\sigma^{\mathrm{DIS}}(x, Q^2, \sqrt{s}) = \sum_{\alpha=q,\bar{q},g} C_\alpha\left(x, \frac{Q^2}{\mu^2}, \sqrt{s}\right) \otimes f_\alpha(x, \mu^2) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right)$$

x : Bjorken-x, Q : momentum transfer, $\sqrt{s}$: collision energy

μ : factorization scale

- Parton Distribution Functions (PDFs)

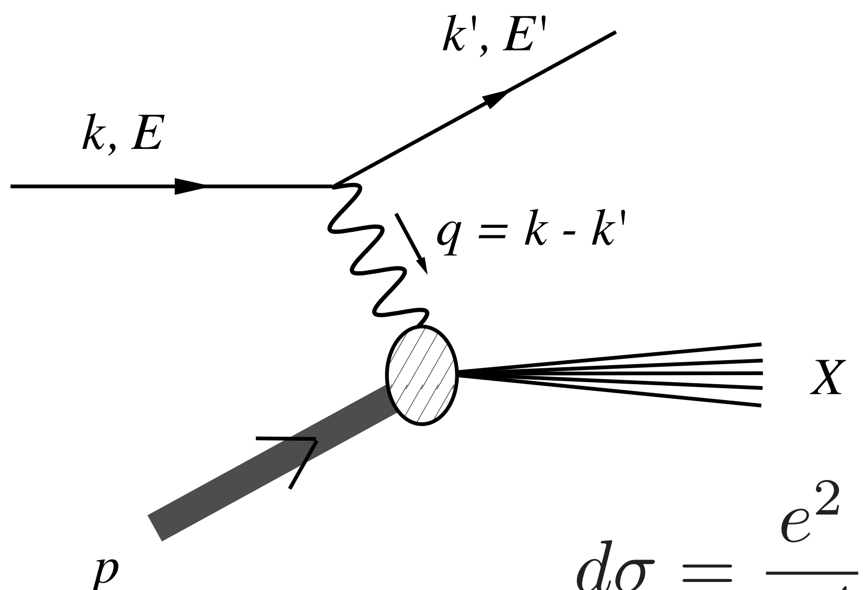
- Probability density for finding a particle with a certain longitudinal momentum fraction x of proton.

- Absorb all perturbative collinear divergences.

- Non-perturbative.

- Universal. → Predictive power of QCD !

PDFs from DIS



$$Q^2 = -q^2, \quad \nu = p \cdot q, \quad x = \frac{Q^2}{2\nu}$$

$$d\sigma = \frac{e^2}{Q^4} \int \frac{d^3 k'}{(2\pi)^3 2E'} \frac{4\pi}{(2E)(2M)} l_{\mu\nu}(k,k') W^{\mu\nu}(p,q)$$

- Hadronic tensor

$$\begin{aligned} W_{\mu\nu}(p,q) &= \frac{1}{4\pi} \sum_X \langle p | j^\dagger_\mu(0) | X \rangle \langle X | j_\nu(0) | p \rangle (2\pi)^4 \delta(p_X - p - q) \\ &= \frac{1}{4\pi} \int d^4 y e^{iq\cdot y} \langle p | [j^\dagger_\mu(y), j_\nu(0)] | p \rangle \end{aligned}$$

- Leptonic tensor

$$l^{\mu\nu}(k,k') = [\bar{u}(k',\sigma')\gamma^\mu u(k,\sigma)]^* \bar{u}(k',\sigma')\gamma^\nu u(k,\sigma)$$

Structure functions

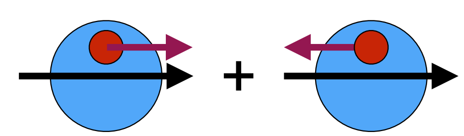
- General form of hadronic tensor

$$
\begin{aligned}
W_{\mu\nu}(p,q,s) &= \frac{1}{4\pi}\int d^4z e^{iqz}\langle p,s|[j_\mu(z),j_\nu(0)]|p,s\rangle \\
&= \left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right)F_1(x,Q^2)+\left(p_\mu-\frac{p\cdot q}{q^2}q_\mu\right)\left(p_\nu-\frac{p\cdot q}{q^2}q_\nu\right)\frac{1}{p\cdot q}F_2(x,Q^2) \\
&\quad +i\epsilon_{\mu\nu\lambda\sigma}q^\lambda s^\sigma\frac{1}{p\cdot q}g_1(x,Q^2)+i\epsilon_{\mu\nu\lambda\sigma}q^\lambda(p\cdot q s^\sigma-s\cdot q p^\sigma)\frac{1}{(p\cdot q)^2}g_2(x,Q^2).
\end{aligned}
$$

structure functions

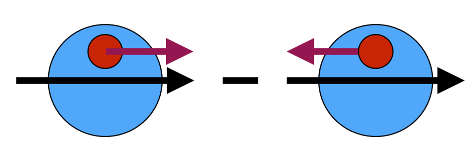
- Structure functions to distribution functions

$$F_\alpha(x,Q^2)=x\sum_f\int_x^1\frac{dz}{z}C_{\alpha f}(z)f\left(\frac{x}{z},\mu^2\right)=x\sum_f C_{\alpha f}(x)\otimes f(x,\mu^2).$$



unpolarized PDFs

$$g_\alpha(x,Q^2)=x\sum_f\int_x^1\frac{dz}{z}E_{\alpha f}(z)\Delta f\left(\frac{x}{z},\mu^2\right)=x\sum_f E_{\alpha f}(x)\otimes\Delta f(x,\mu^2).$$



polarized PDFs

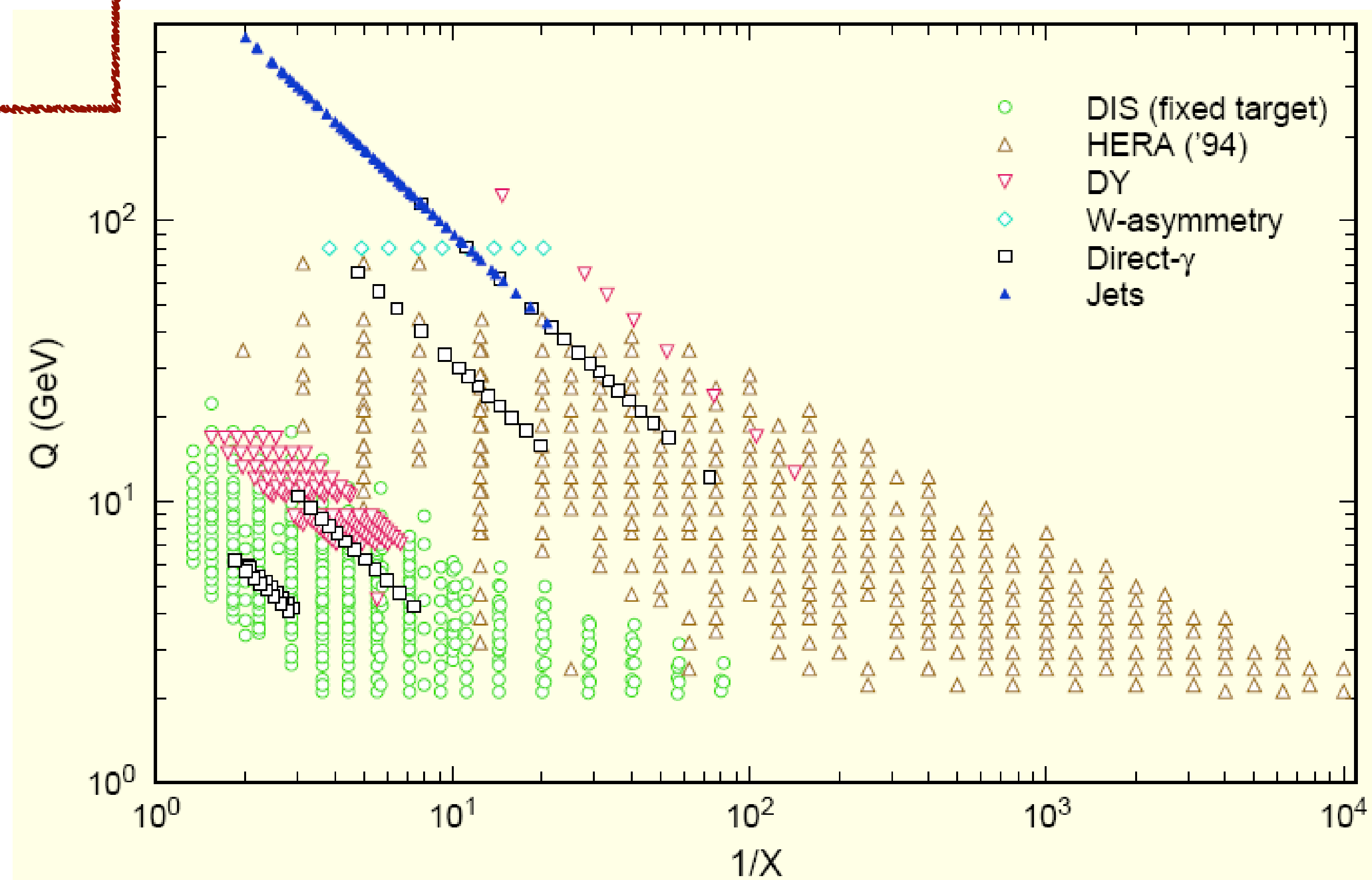
Global QCD analysis

- Extract PDFs from experiment data

PDFs at Q_0 with initial fit params
$f_\alpha(x, Q_0)$

change fit params to minimize χ^2

χ^2



DGLAP evolution

PDFs at $Q > Q_0$
$f_\alpha(x, Q)$

PT QCD calculation
$C_\alpha(x, 1, \sqrt{s})$

cross section
$\sigma(x, Q^2, \sqrt{s})$

Global QCD analysis

Many ongoing or planned experiments

(BNL, JLab, J-PARC, COMPASS, GSI, EIC, LHeC, …)

- Choices made for the analysis
  - Data sets and kinematic cuts
  - Strong coupling constant
  - Parametrization of the PDFs

$$f(x, Q_0, \{a_i\}) = x^{a_1}(1-x)^{a_2}C(x, \{a_i\})$$

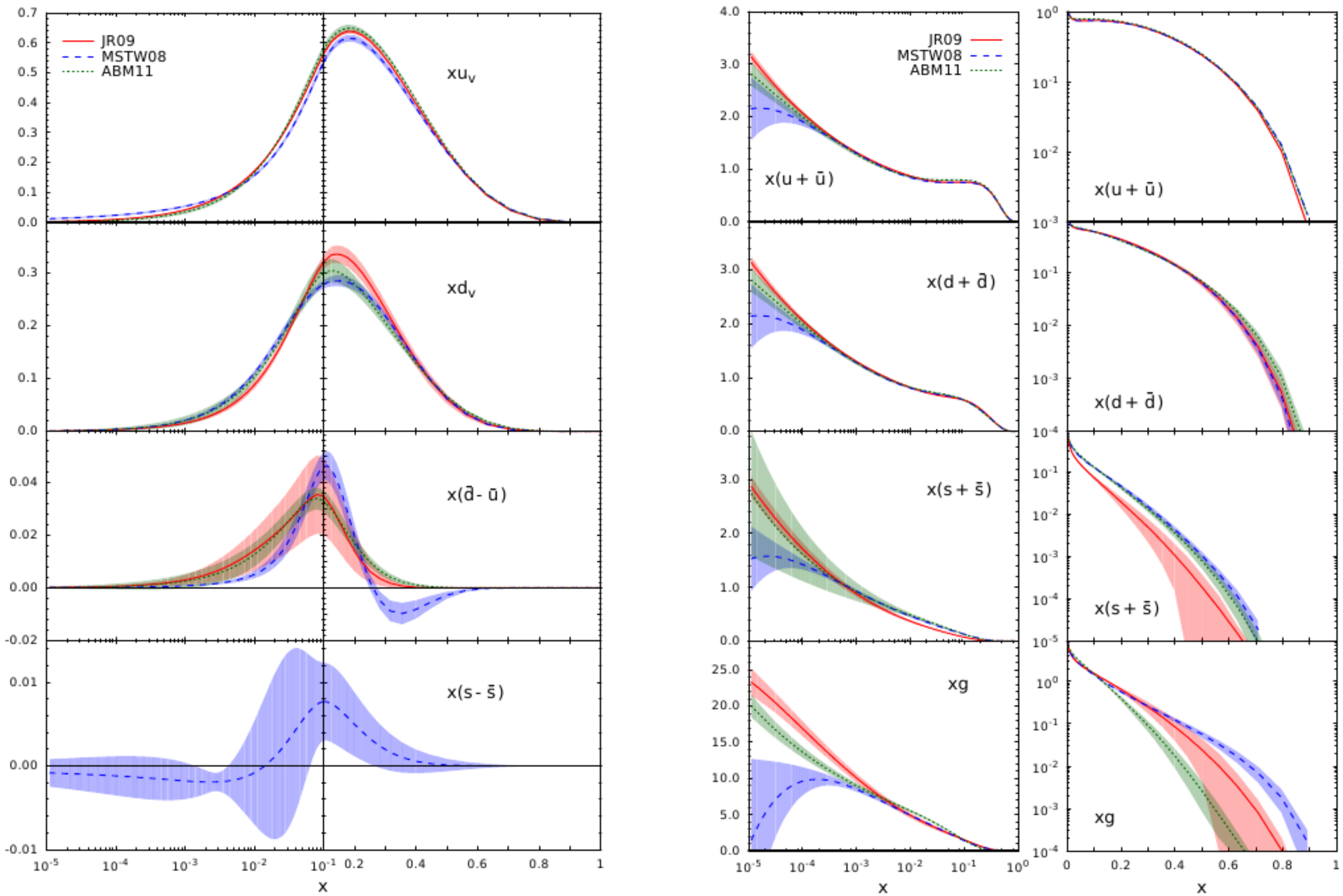
$C(x, \{a_i\})$: interpolation function

20~40 free parameters

- Discrepancies between analyses
- Currently dominant errors in Higgs production
- Precise PDFs are needed for new physics

Global QCD analysis

- Uncertainties of PDFs



[Jimenez-Delgado et al.(2013)]

Large uncertainties in both small and large x region.

Global QCD analysis

- Large-x

Testing ground for models of hadron structure

- SU(6) spin-flavor symmetry

$$d/u \longrightarrow 1/2$$

- Scalar diquark dominance

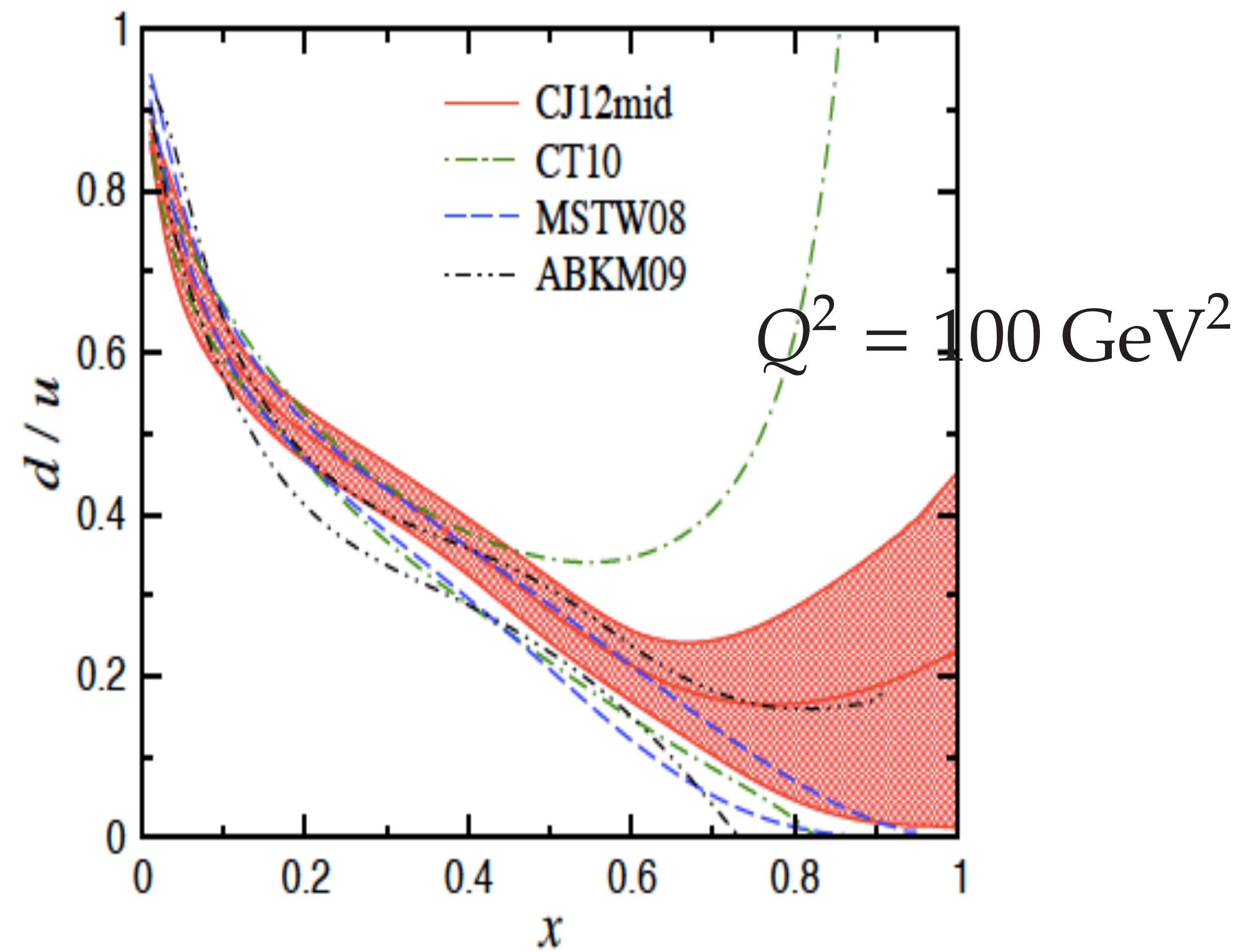
$$d/u \longrightarrow 0$$

- pQCD power counting

$$d/u \longrightarrow 1/5$$

- Local quark-hadron duality

$$d/u \longrightarrow 0.42$$



[Jimenez-Delgado et al.(2013)]

PDFs from lattice

- Quark distribution by light-cone operator

$$q(x,\mu) = \int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \langle \mathcal{N}(P)|O(\xi^-)|\mathcal{N}(P)\rangle,$$
$$O(\xi^-) = \overline{\psi}(\xi^-)\gamma^+ U_+(\xi^-,0)\psi(0)$$

- $\xi^\pm = (t \pm z)/\sqrt{2}$: light-cone coordinate
- Time-dependent. $\Longrightarrow$ Not calculable on the lattice directly.

- Moments

$$a_n = \int_0^1 dx x^{n-1} q(x) = \frac{1}{P^{\mu_1}\ldots P^{\mu_n}} \langle \mathcal{N}(P)|O^{\{\mu_1\cdots\mu_n\}}|\mathcal{N}(P)\rangle$$
$$O^{\{\mu_1\cdots\mu_n\}} = \overline{\psi}(0)\gamma^{\{\mu_1} i\overleftrightarrow{D}^{\mu_2}\ldots i\overleftrightarrow{D}^{\mu_n\}}\psi(0)$$

- Written in local operators. Calculable on lattice (in principle).
- But, higher moments are difficult to be accessed.

Quasi distribution approach

[X.-D. Ji (2013)]

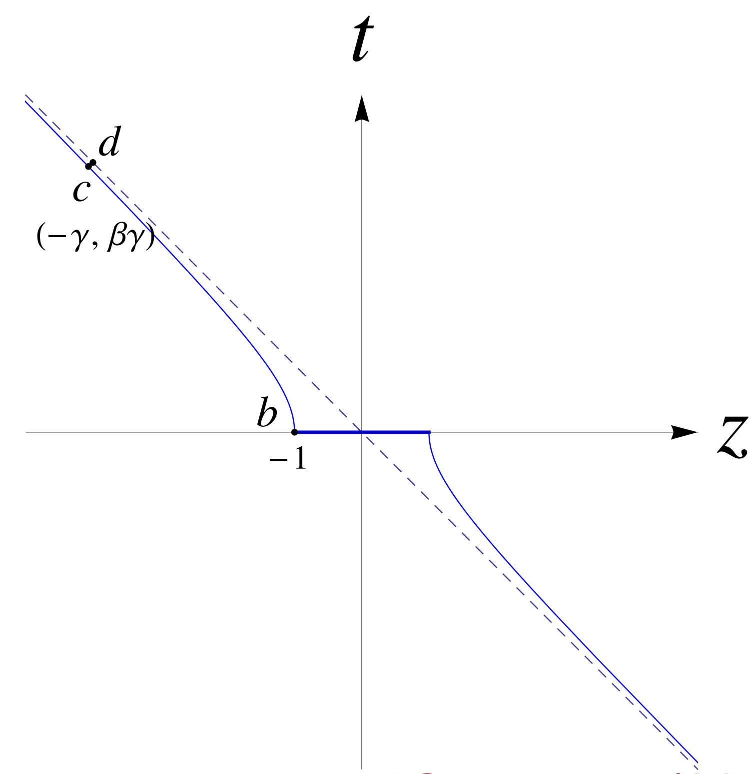
Light-cone distributions

$$
q(x,\mu) = \int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \langle \mathcal{N}(P)|O(\xi^-)|\mathcal{N}(P)\rangle,
$$
$$
O(\xi^-) = \overline{\psi}(\xi^-)\gamma^+ U_+(\xi^-,0)\psi(0)
$$

- $\xi^\pm = (t \pm z)/\sqrt{2}$: light-cone coordinate
- Boost invariant in the z-direction
- Quark fields are on the light-cone.

Quasi distribution approach

- Off the light-cone
  - Slightly off the light-cone
  - Go to rest frame
  - Proton is moving with infinite momentum
  - Quark bilinear sits on z-axis (spatial).
  - Equal time operator, lattice calculable.



[Chen et.al.(2016)]

- Quasi distributions [X.-D. Ji (2013)]

$$\widetilde{q}(\tilde{x}, \mu, P_z) = \int \frac{d\delta z}{2\pi} e^{-i\tilde{x}P_z\delta z} \langle \mathcal{N}(P_z)|\widetilde{O}(\delta z)|\mathcal{N}(P_z)\rangle,$$
$$\widetilde{O}(\delta z) = \overline{\psi}(\delta z)\gamma^z U_z(\delta z, 0)\psi(0)$$

  - Sit in spatial direction. Time-independent.
  - Boost variant in the z-direction

Quasi-PDFs [Ji (2013)]

- Quasi distributions

$$\widetilde{q}(\tilde{x},\mu,P_z)=\int\frac{d\delta z}{2\pi}e^{-i\tilde{x}P_z\delta z}\langle\mathcal{N}(P_z)|\widetilde{O}(\delta z)|\mathcal{N}(P_z)\rangle,$$

$$\widetilde{O}(\delta z)=\overline{\psi}(\delta z)\gamma^z U_z(\delta z,0)\psi(0)$$

- Separated in spatial z-direction. Calculable on lattice.
- In the limit of $P_z\to\infty$, normal distributions are recovered.

- Matching (Large Momentum Effective Theory)

$$\widetilde{q}(x,\Lambda,P_z)=\int\frac{dy}{y}Z\left(\frac{x}{y},\frac{\Lambda}{P_z},\frac{\mu}{P_z}\right)q(y,\mu)+\mathcal{O}\left(\frac{\Lambda_{\rm QCD}^2}{P_z^2},\frac{M^2}{P_z^2}\right)$$

- Z can be perturbatively obtained. one-loop [Xiong et al. (2014)]
- Large P_z is required for small corrections.

QCD collinear factorization approach

[Ma and Qiu (2014)]

- Going back to the collinear factorization

$$\sigma^{\mathrm{DIS}}(x, Q^2, \sqrt{s}) = \sum_{\alpha=q,\bar{q},g} C_\alpha\left(x, \frac{Q^2}{\mu^2}, \sqrt{s}\right) \otimes f_\alpha(x, \mu^2) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{Q^2}\right)$$

All CO divergences are factorized into the PDFs with PT hard coefficients.

- Lattice calculable cross section

$$\widetilde{\sigma}(x, \widetilde{\mu}^2, P_z) = \sum_{\alpha=q,\bar{q},g} \widetilde{C}_\alpha\left(x, \frac{\widetilde{\mu}^2}{\mu^2}, P_z\right) \otimes f_\alpha(x, \mu^2) + \mathcal{O}\left(\frac{\Lambda^2_{\mathrm{QCD}}}{\widetilde{\mu}^2}\right)$$

All CO divergences are factorized into the PDFs with PT hard coefficients.

$\mu \longleftrightarrow \mu$ (factorization scale)

$Q \longleftrightarrow \widetilde{\mu}$ (resolution)

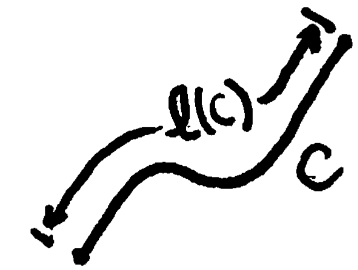
$\sqrt{s} \longleftrightarrow P_z$ (parameter)

Quasi-PDFs = the lattice calculable cross section.
Factorizable to all-orders.

Renormalization

- Renormalization of non-local quark bilinear

$$O_{\mathcal{C}} = Z_{\psi,z} e^{\delta m \ell(\mathcal{C})} O_{\mathcal{C}}^{\text{ren}}$$



- $Z_{\psi,z}$: ψ , z-field wave function, ψ -z-field vertex renormalization
- (multiplicative) renormalizability has been proven up to two-loop.

  (Now, all-order proof has been reported.) [T. I. et al (2017), Ji et al. (2017)]

- The existence of the continuum limit for the HQET has been confirmed in the lattice QCD simulations. (numerical NPT proof)

- Power divergence
- Power divergence makes the theory ill-defined.

  (e.g. no continuum limit on lattice.)

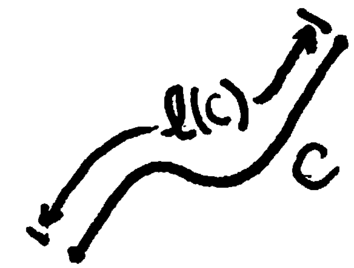
- The power divergence must be subtracted nonperturbatively.
- Power divergence subtracted non-local operator:

$$\widetilde{O}^{\text{subt}}(\delta z) = e^{-\delta m |\delta z|} \widetilde{O}(\delta z)$$

Renormalization

- Renormalization of Wilson lines

$$W_{\mathcal{C}} = Z_z e^{\delta m \ell(\mathcal{C})} W_{\mathcal{C}}^{\text{ren}}$$



- Well-known. [Dotsenko, Vergeles, Arefeva, Craigie, Dorn, … ('80)]
- δm : mass renormalization of a test particle moving along $\mathcal{C}$

All the power divergence is contained.

- Auxiliary z-field (just like static heavy quark)

- By integrating out the z-field, the Wilson line is recovered.

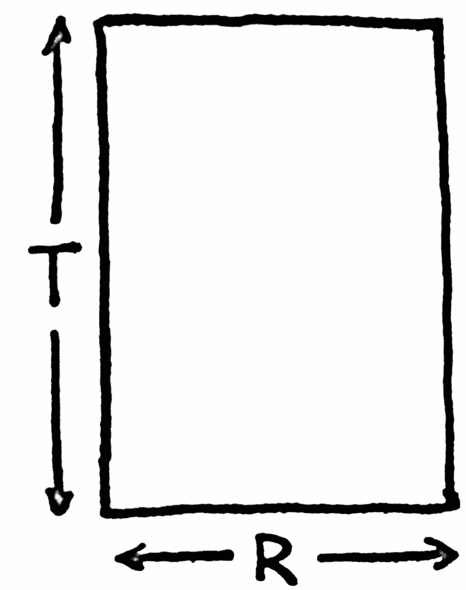
$$\int \mathcal{D}\bar{z}\mathcal{D}z e^{-\int_x \bar{z}(D_z+m)z} z(\delta z)\bar{z}(0) = \langle z(\delta z)\bar{z}(0)\rangle = U_z(\delta z, 0)$$

- Additive mass renormalization δm
- z-field wave function renormalization Z_z

Subtracting power divergences

- Choice of δm [Musch et al. (2011), T.I. et al. (2016)]
  - One way is to use static $Q\bar{Q}$ potential $V(R)$.
  - $V(R)$ is obtained from Wilson loop:

$$W_{R\times T} \propto e^{-V(R)T} \quad (T \to \text{large})$$



  - Renormalization of $V(R)$:

$$V^{\text{ren}}(R) = V(R) + 2\delta m$$

  - Renormalization condition (fix a renormalized quantity) :

$$V^{\text{ren}}(R_0) = V_0 \longrightarrow \delta m = \frac{1}{2}(V_0 - V(R_0))$$

- Power divergence free quasi distributions

$$\boxed{\tilde{q}^{\text{subt}}(\tilde{x}, \mu, P_z) = \int \frac{d\delta z}{2\pi} e^{-i\tilde{x}P_z\delta z} e^{-\delta m|\delta z|} \langle \mathcal{N}(P_z)|\widetilde{O}(\delta z)|\mathcal{N}(P_z)\rangle}$$

Nonperturbative renormalization

- Regularization Independent Momentum Scheme

Renormalization condition

$$Z_\Gamma^{\mathrm{RI/MOM}(\mu_R)} \langle p|O_\Gamma|p\rangle\Big|_{p^2=\mu_R^2} = \langle p|O_\Gamma|p\rangle_0\big|_{p^2=\mu_R^2} .$$

$$S_f(p)\big|_{p^2=\mu_R^2} = Z_f^{\mathrm{RI/MOM}(\mu_R)} S_f^0(p)\Big|_{p^2=\mu_R^2}$$

- Calculate matrix elements of the operator between off-shell single particle momentum eigenstates in interacting and free case.

- Regularization independent (lattice, dimensional, whatever)

- Nonperturbative Z can be obtained by lattice simulation.

- Matching to MSbar is calculable perturbatively with dimensional regularization (easy to manage to go to higher-loop).

Operator mixing

- Local quark bilinear

$$O_\Gamma = \overline{\psi}(x)\Gamma\psi(x)$$

$$Z_\Gamma \langle O_\Gamma \rangle^R = \langle O_\Gamma \rangle.$$

no mixing.

- Non-local quark bilinear

[Constantinou et.al. (2017)] (PT)

[Chen et.al.(2017)] (NPT proof)

$$O_\Gamma(\delta z) = \overline{\psi}(x+\delta z)\Gamma U_z(x+\delta z; x)\psi(x)$$

- e.g. unpolarized case with Wilson fermion

$$\boxed{\begin{pmatrix} Z_{V_zV_z}(z) & Z_{V_z\mathcal{I}}(z) \\ Z_{\mathcal{I}V_z}(z) & Z_{\mathcal{I}\mathcal{I}}(z) \end{pmatrix} \begin{pmatrix} \langle O_{\gamma_z}(z)\rangle^R \\ \langle O_{\mathcal{I}}(z)\rangle^R \end{pmatrix} = \begin{pmatrix} \langle O_{\gamma_z}(z)\rangle \\ \langle O_{\mathcal{I}}(z)\rangle \end{pmatrix},}$$

$$Z_{V_i}(z)\langle O_{\gamma_{i\neq z}}(z)\rangle^R = \langle O_{\gamma_{i\neq z}}(z)\rangle.$$

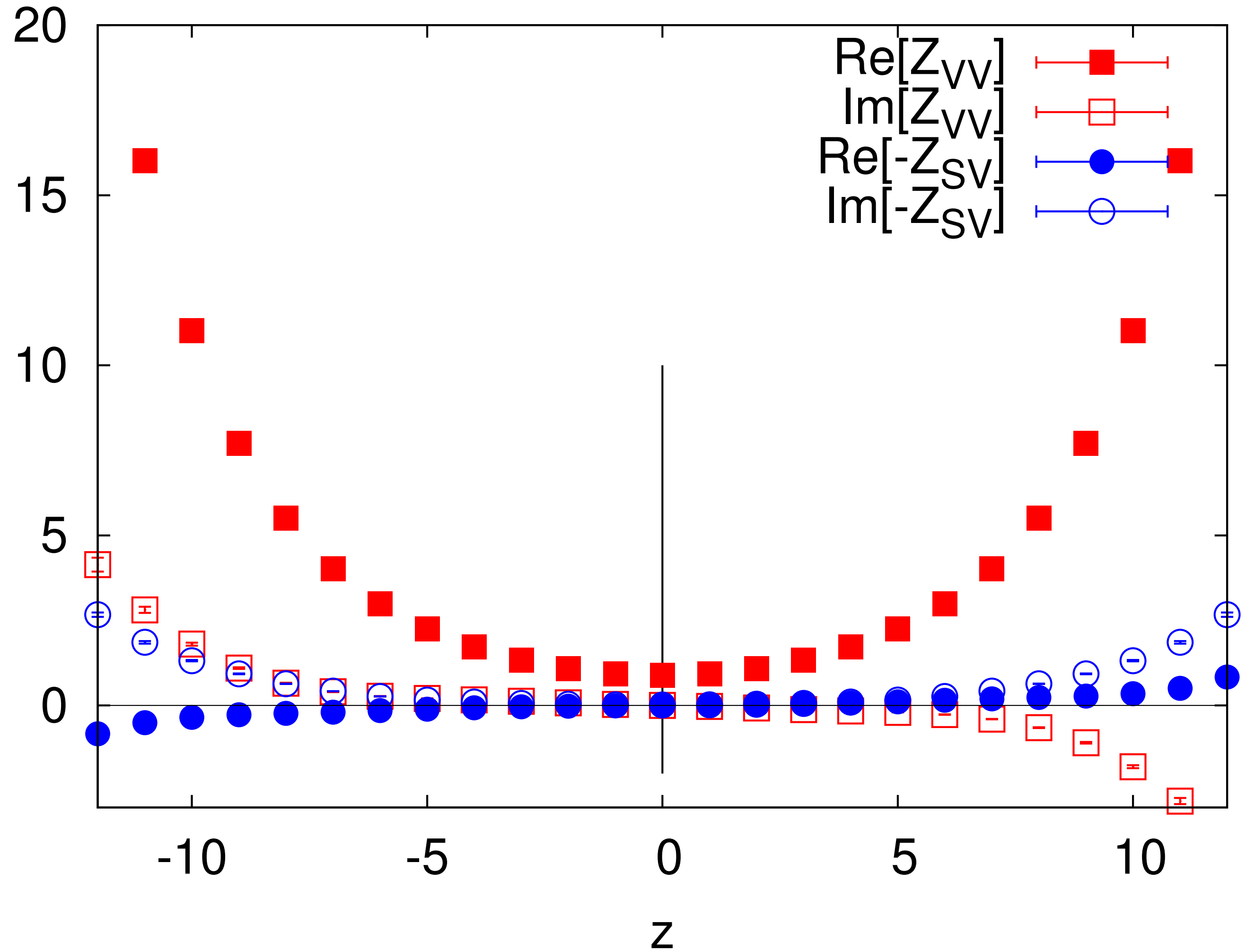
- e.g. unpolarized case with chiral fermion

$$Z_V(z)\langle O_{\gamma_\mu}(z)\rangle^R = \langle O_{\gamma_\mu}(z)\rangle.$$

$\Gamma = \gamma_t$ could be used for quasi-PDF to avoid the mixing.

Lattice renormalization factor

- NPT RI/MOM renorm factor [Chen et.al.(2017)]



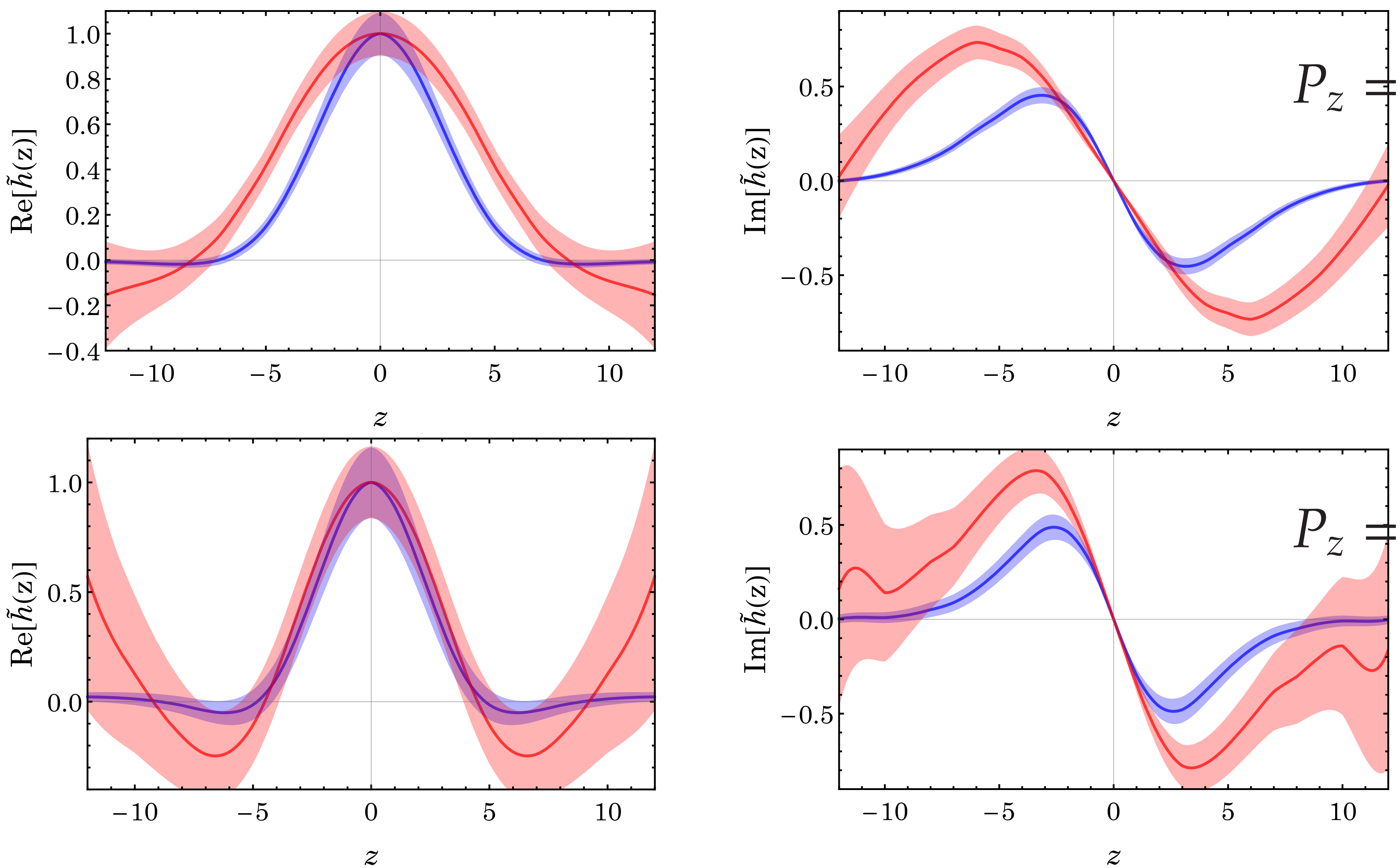
$$P_z = 1.29 \text{ GeV}$$
$$\mu_R = 2.4 \text{ GeV}$$
$$1/a = 1.64 \text{ GeV}$$

- Huge value in the large z region is due to power divergence.
- Mixing effect might be small.

Renormalized matrix element

- RI/MOM renormalized proton matrix element [Chen et.al.(2017)]

$$h^{\mathrm{RI/MOM}(\mu_R)}(z,P_z)=Z^{\mathrm{RI/MOM}(\mu_R)}_{\mathrm{LAT}(1/a)}(z,P_z)h^{\mathrm{LAT}(1/a)}(z,P_z)$$



$$P_z=\frac{4\pi}{L}\sim 0.86\ \mathrm{GeV}$$

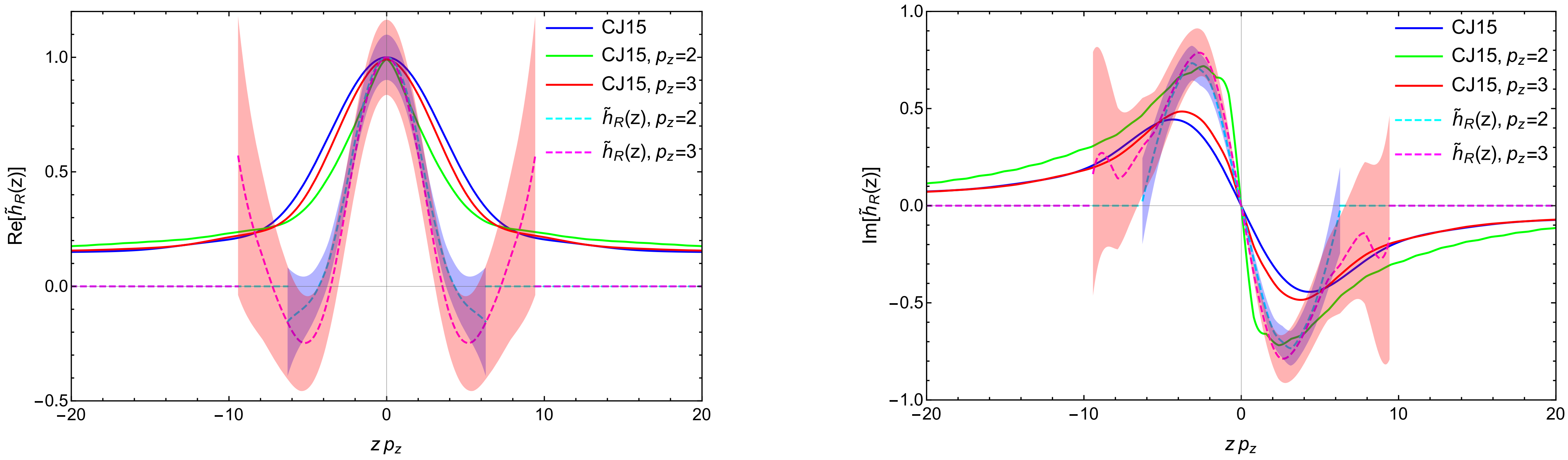
$$P_z=\frac{6\pi}{L}\sim 1.29\ \mathrm{GeV}$$

$$\mu_R=2.4\ \mathrm{GeV}$$

Amplitude in the large-z region is magnified.

Renormalized matrix element

- Global QCD analysis results back to quasi-matrix element [Chen et.al.(2017)]



(1) Matching: CJ15 —> quasi-PDF

(2) Fourier Transform: quasi-PDF —> matrix element

Long tail is expected from the global QCD analysis result.

LC PDF with NPT renormalization

- Cutoff on z

[Chen et al.(2017)]

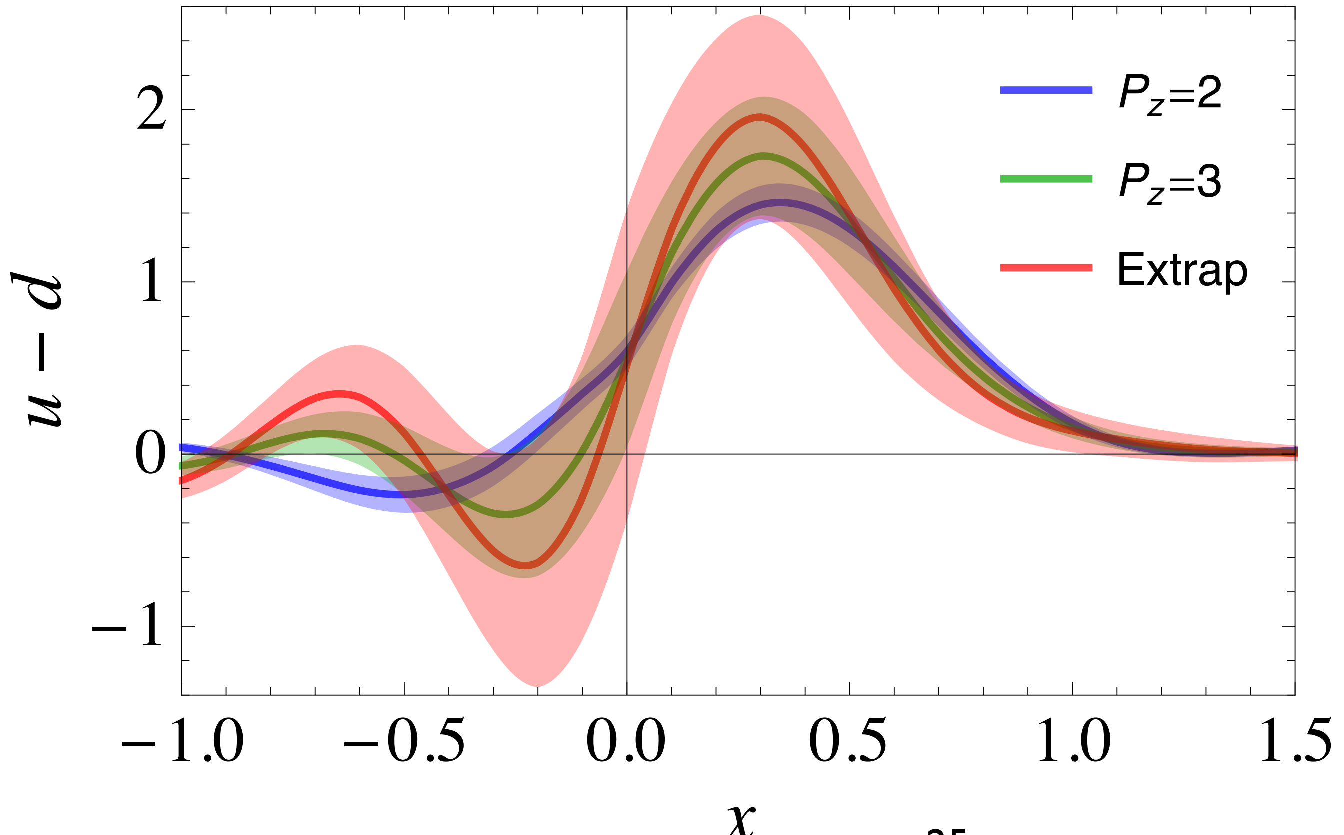
Cutoff (|z|=12) has to be introduced in the Fourier transform.

The cutoff did not give any significant effect in the previous study, because bare matrix elements are well-suppressed in large z.

- RI/MOM renormalized quasi to MSbar LC PDF

$$q^{\overline{\mathrm{MS}}(\mu)}(x) = \int_{-\infty}^{\infty} \frac{dy}{|y|} \boxed{C^{\overline{\mathrm{MS}}(\mu)}_{\mathrm{RI/MOM}(\mu_R)}\left(\frac{x}{y}, \frac{\mu_R}{P_z}, \frac{\mu}{P_z}\right)} \tilde{q}^{\mathrm{RI/MOM}(\mu_R)}(y, P_z) + O\left(\frac{\Lambda^2_{\mathrm{QCD}}}{P_z^2}\right)$$

one-loop [I. Stewart and Y. Zhao (2017)]



Oscillation problem.

LC PDF with NPT renormalization

- Other group (ETMC)

[Alexandrou et al.(2017)]

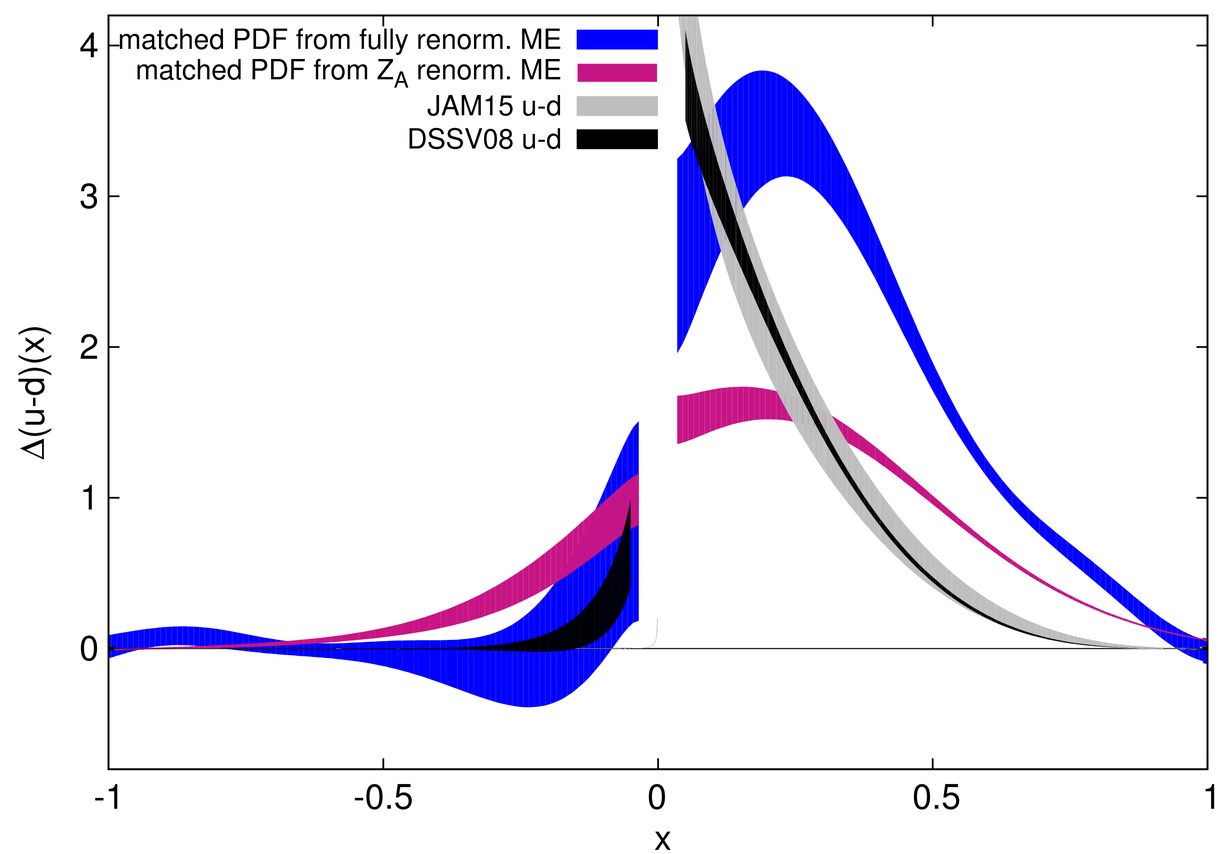


Fig. 10. Comparison of matched helicity PDF obtained from quasi-PDF computed with either fully renormalized matrix elements (blue) or with bare matrix elements multiplied by the local ($z = 0$) axial current Z-factor, Z_A (magenta). For purely orientational purposes, we also plot phenomenological PDFs (DSSV08 [51] and JAM15 [52]). However, we emphasize that no quantitative comparison with our results is aimed at, since careful consideration of a number of systematic effects is still needed. These include: cut-off effects, non-physical pion mass, finite volume effects, possible contamination by excited states, extrapolation to infinite nucleon boost, as well as the improvements in the computation of $\overline{\text{MS}}$ renormalization functions, postulated in the previous subsection.

Reducing the oscillation

- Filter and derivative method

[Lin et al.(2017)]

derivative method

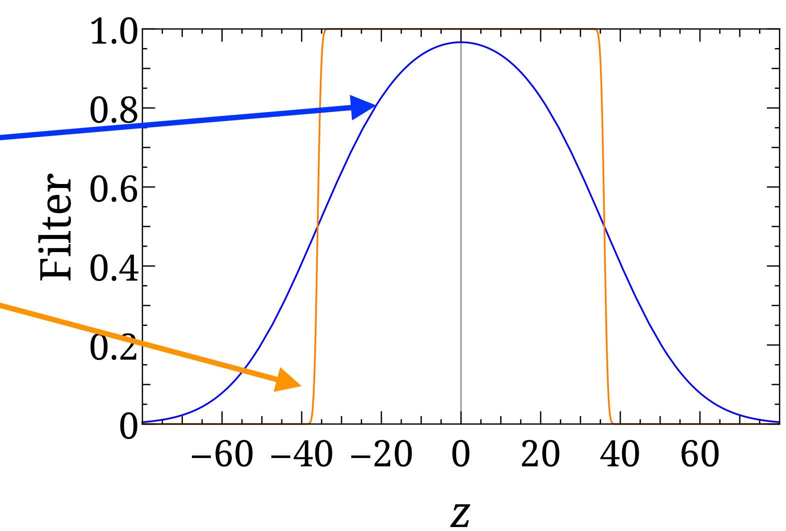
$$q(x) = -\int_{-z_{\max}}^{+z_{\max}} \frac{dz}{2\pi} \frac{e^{ixP_z z}}{iP_z x} h'(z)$$

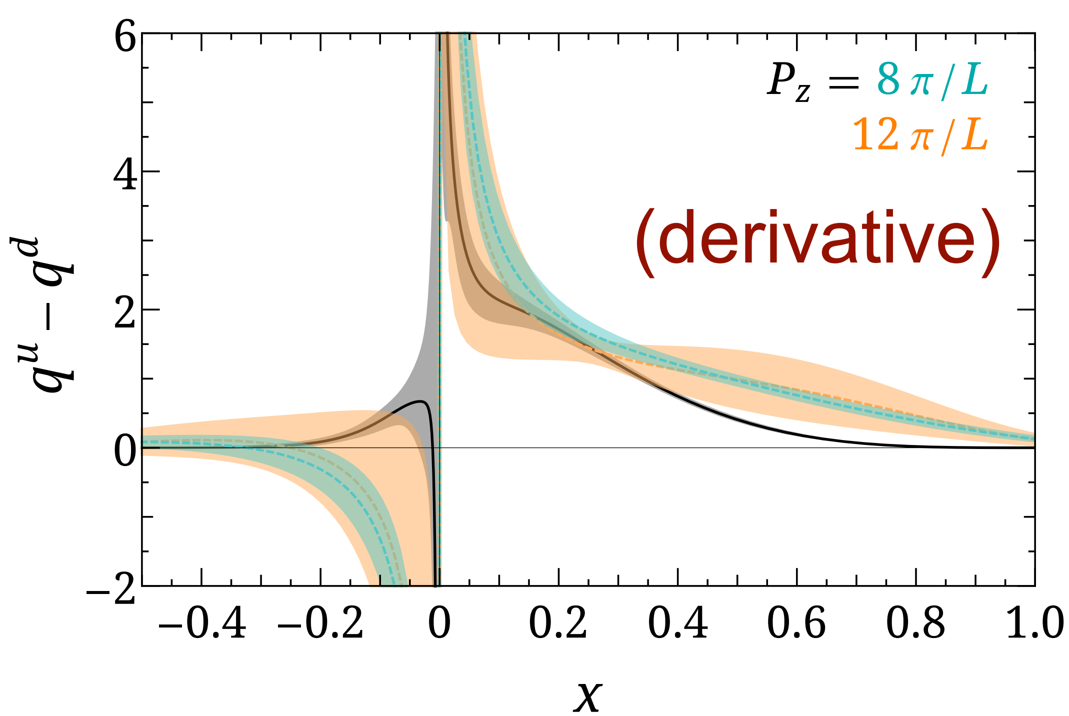
filter using sigmoidal error function

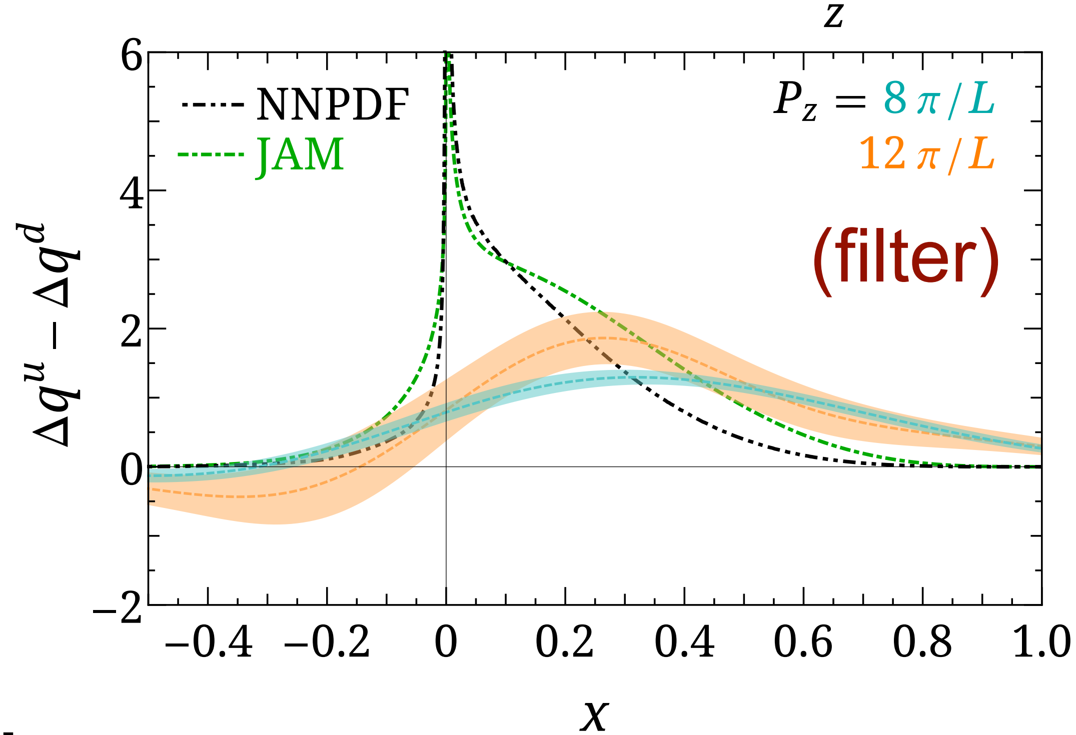
$$F(z_{\mathrm{lim}}, z_{\mathrm{wid}}) = \frac{1+\mathrm{erf}\left(\frac{z+z_{\mathrm{lim}}}{z_{\mathrm{wid}}}\right)}{2} \frac{1-\mathrm{erf}\left(\frac{z-z_{\mathrm{lim}}}{z_{\mathrm{wid}}}\right)}{2}$$

$$\{z_{\mathrm{lim}}, z_{\mathrm{wid}}\} = \{36, 24\}$$

$$\{z_{\mathrm{lim}}, z_{\mathrm{wid}}\} = \{32, 1\}$$







Renormalization with auxiliary field

Auxiliary field (like static quark)

[Ji et.al.(2017), Green et al.(2017)]

$$\boxed{\begin{aligned} S_\zeta &= \int_z \overline{\zeta}(x+z)\,(D_z + m_0)\,\zeta(x+z) \\ \langle \zeta(x+z)\overline{\zeta}(x)\rangle_\zeta &= \theta(z)e^{-m|z|}U_z(x+z,x) \end{aligned}}$$

$$\phi(z) = \overline{\zeta}(z)\psi(z)$$

$$O_\Gamma(z) = \overline{\psi}(z)\Gamma U_z(z,0)\psi(0) \longrightarrow O_\Gamma(z) = \langle \overline{\phi}(z)\Gamma\phi(0)\rangle_\zeta$$

Renormalization

- $\phi(z)$ is a local operator. Renormalized locally.

$$O_\Gamma^R(z) = Z_\phi^2 e^{-m|z|} O_\Gamma(z)$$

- Self-energy of auxiliary field can be identified to Wilson line's self-energy.
- It can be related to static-heavy light current renormalization.

Renormalization with auxiliary field

Numerical results

[Green et al. (2017)]

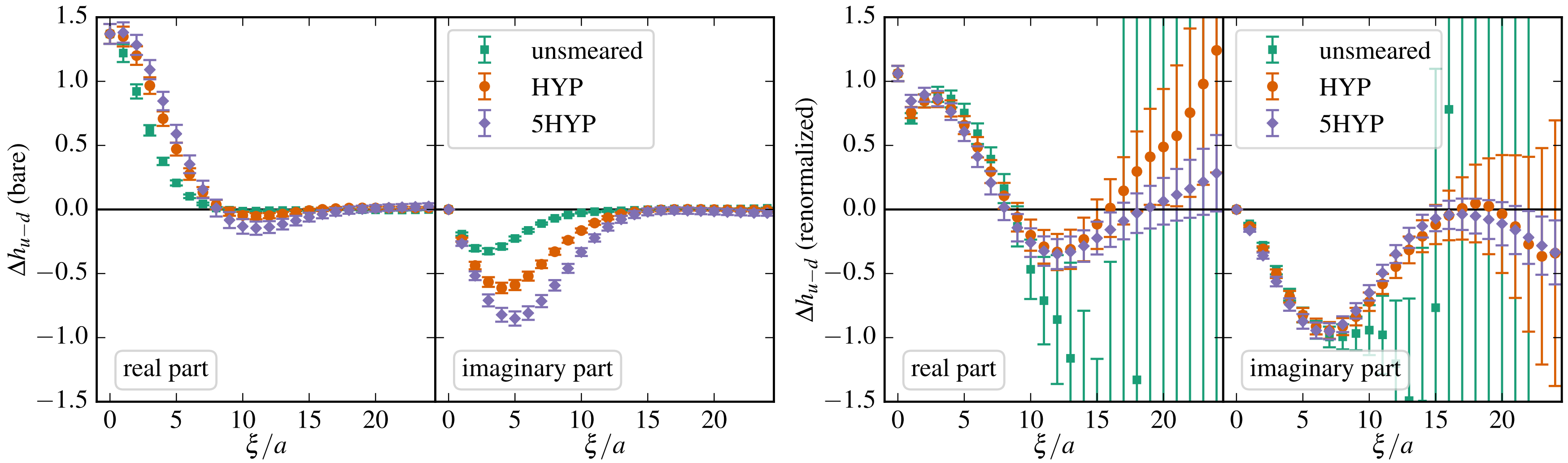


FIG. 3. Matrix element for the helicity quasi-PDF versus ξ on the $\beta = 2.10$ ensemble, for three different link discretizations, bare (left) and renormalized (right). Only $\xi \geq 0$ is shown, since the real part is even in ξ and the imaginary part is odd.

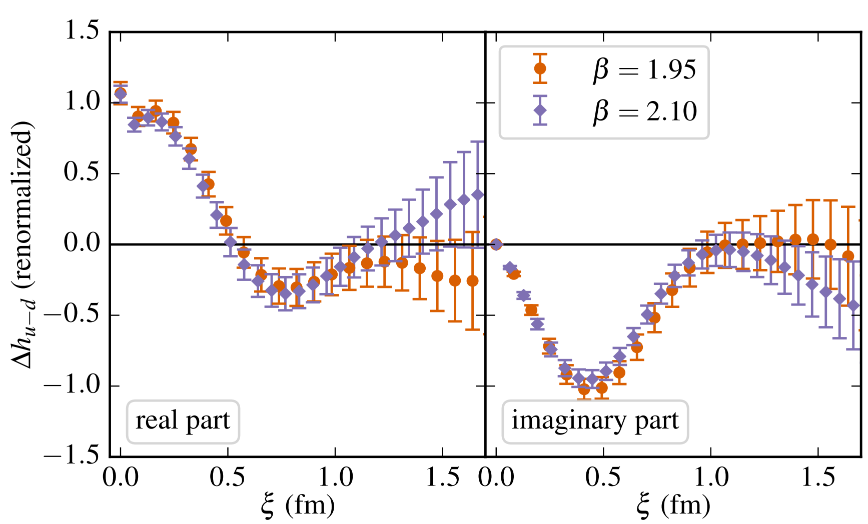


FIG. 4. Renormalized matrix element for the helicity quasi-PDF versus ξ on the two ensembles, using five steps of HYP smearing.

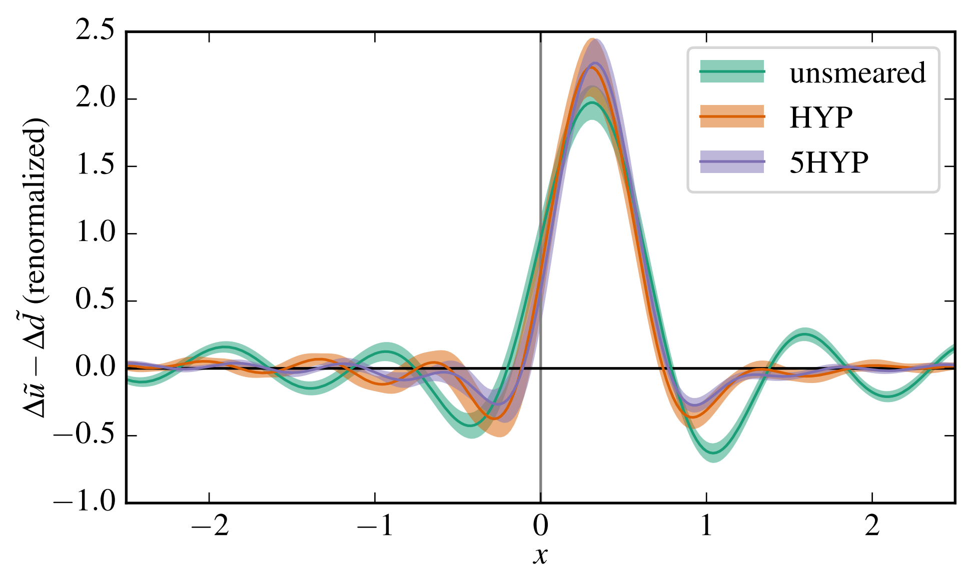


FIG. 5. Isovector helicity quasi-PDF on the $\beta = 2.10$ ensemble, for three different link discretizations, computed from renormalized matrix elements.

Pseudo-PDFs [A. Radyushkin (2017)]

- Lorentz decomposition of matrix element

$$\begin{aligned}\mathcal{M}^\alpha(z,p) &= \langle p|\overline{\psi}(z)\gamma^\alpha W_z(z,0)\psi(0)|p\rangle \\ &= 2p^\alpha \mathcal{M}_p(-(zp),-z^2) + z^\alpha \mathcal{M}_z(-(zp),-z^2).\end{aligned}$$

- Light-cone

$$p = (p_+, 0, 0_\perp), \quad z = (0, z_-, 0_\perp)$$

$$\mathcal{M}^+(z,p) = 2p^+ \mathcal{M}_p(-p_+z_-, 0)$$

$$\mathcal{M}_p(-p_+z_-, 0) = \int_{-1}^{1} dx e^{-ixp_+z_-} f(x)$$

light-cone PDF

Pseudo-PDFs

- Ioffe time PDF

$\mathcal{M}_p(-zp, -z^2)$ Lorentz invariant. Computable in any frame.

$\nu = -pz$ Ioffe time [B. L. Ioffe (1969)]

$$\mathcal{M}_p(\nu, -z^2) = \int_{-1}^{1} dx e^{ix\nu} \mathcal{P}(x, -z^2).$$

Ioffe time PDF pseudo-PDF

Pseudo-PDF has $-1 \leq x \leq 1$ support. [A. Radyushkin (2017)]

- $z^2 \to 0$ limit

$$\mathcal{M}_p(\nu, 0) = \int_{-1}^{1} dx e^{ix\nu} f(x) \qquad \left(\mathcal{M}_p(-p_+ z_-, 0) = \int_{-1}^{1} dx e^{-ixp_+ z_-} f(x)\right)$$

$$\mathcal{P}(x, -z^2) \xrightarrow[z^2 \to 0]{} f(x)$$

Pseudo-PDFs

- Quasi-PDF case

$$p = (E, 0_\perp, p_3), \quad z = (0, 0_\perp, z_3)$$

$$\mathcal{M}^3(z,p) \quad = \quad 2p^3 \mathcal{M}_p(-z_3 p_3, -z_3^2) + z^3 \mathcal{M}_z(-z_3 p_3, -z_3^2).$$

$$\begin{aligned}\widetilde{q}(\tilde{x}, p_3) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dz e^{-i\tilde{x}p_3 z}\mathcal{M}^3(z,p) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu e^{-i\tilde{x}\nu}\left[\mathcal{M}_p(\nu, \nu^2/p_3^2) - \frac{\nu}{2p_3^2}\mathcal{M}_z(\nu, \nu^2/p_3^2)\right].\end{aligned}$$

$$\widetilde{q}(x, p_3) \xrightarrow[p_3 \to \infty]{} f(x)$$

- Better choice

$$\mathcal{M}^0(z,p) \quad = \quad 2p^0 \mathcal{M}_p(-z_3 p_3, -z_3^2).$$

$$\widetilde{q}'(\tilde{x}, p_3) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dz e^{-i\tilde{x}p_3 z}\mathcal{M}^0(z,p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu e^{-i\tilde{x}\nu}\mathcal{M}_p(\nu, \nu^2/p_3^2).$$

Pseudo-PDFs

- Ratio

$$\mathfrak{M}(\nu, z_3^2) = \frac{\mathcal{M}_p(\nu, z_3^2)}{\mathcal{M}_p(0, z_3^2)}$$

$$\mathcal{M}_p(0, z_3^2) \xrightarrow[z_3^2 \to 0]{} 1$$ regular in the limit

By taking the ratio:
 - smaller scaling violation in $z_3 \to 0$
 - power divergence is canceled and well defined in taking continuum limit

- Scale evolution (DGLAP)

different z_3^2 ⟷ different scale

$$\frac{d}{d \ln z_3^2} \mathcal{M}(\nu, z_3^2) = -\frac{\alpha_s}{2\pi} C_F \int_0^1 du B(u) \mathcal{M}(u\nu, z_3^2), \quad B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

Pseudo-PDFs

Numerical results

[Orginos et al. (2017)]

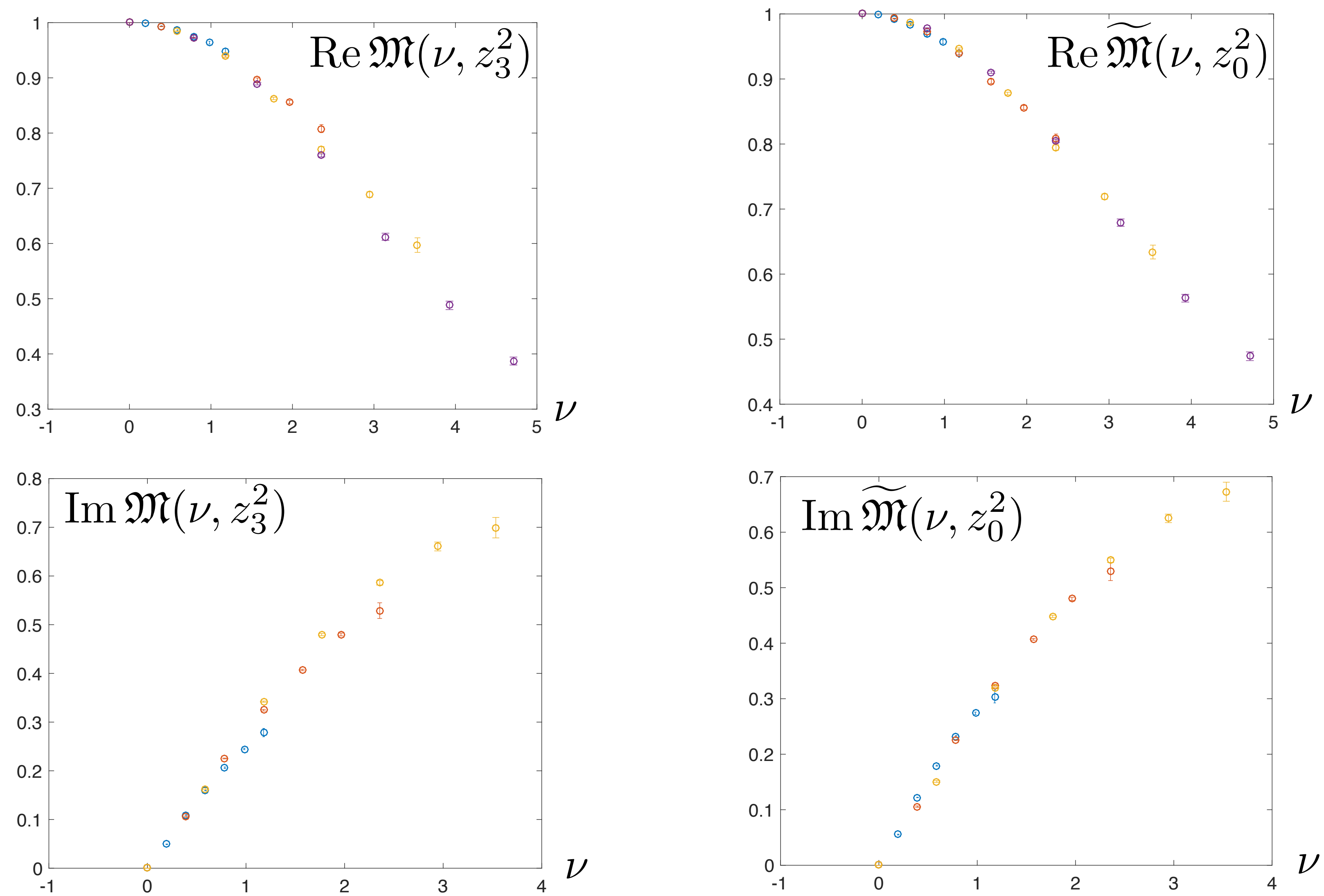


Figure 3. The ratio $\mathfrak{M}(\nu, z_3^2)$ for for $z_3/a = 1, 2, 3$, and 4. **: Top:** Real part. **Bottom:** Imaginary part. **Left:** Data before evolution. **Right:** Data after evolution. The reduction in scatter indicates that evolution collapses all data to the same universal curve.

evolved to

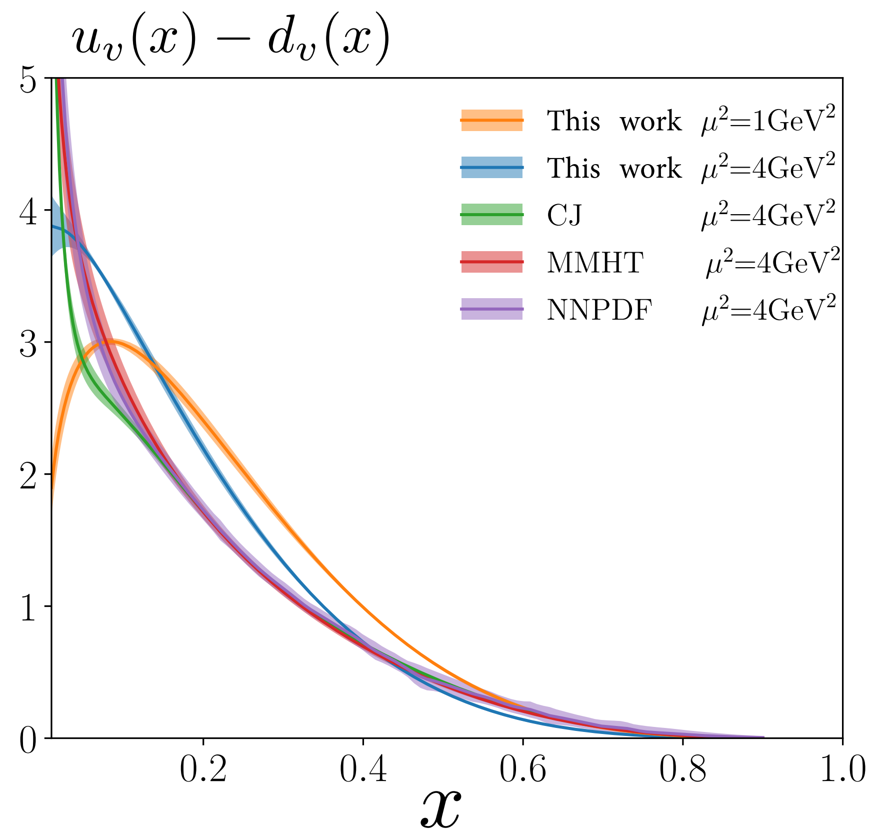
$$1/z_0 = 1\ \mathrm{GeV}$$

By the scale evolution, data are nicely aligned.

Pseudo-PDFs

Numerical results

[Orginos et al. (2017)]



Functional form is assumed.

$$q_v(x) \sim x^a(1-x)^b$$

Pseudo v.s. Quasi

- By taking the ratio, pseudo-PDF is better for renormalization.
- Small-x region requires large $\nu = -pz$; eventually large momentum data is required (?)

$$\mathcal{M}_p(\nu, -z^2) = \int_{-1}^{1} dx e^{ix\nu} \mathcal{P}(x, -z^2).$$

Summary and outlook

- New approach for lattice calculation of PDFs has been proposed:

  Quasi-PDFs [X.-D. Ji (2013)]

- Many tools have been developed:
  - Renormalization (perturbative, nonperturbative)
  - Treatment of power divergence
  - High-momentum smearing technique
- Similar approach to quasi-PDFs: Pseudo- PDFs [A. Radyushkin(2013)]
- Some other approaches: Hadronic tensor [K.F. Liu et al. (1997-2017)]

  Compton amplitude [A.J. Chambers et al. (2017)]
- Lattice QCD could help global QCD analysis.
- Transverse momentum dependent parton densities (TMDs) and Generalized parton distributions (GPDs) could also be addressed toward full scan of 3D structure of nucleons.